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ABSTRACT

This document contains plenary and contributed papers presented at the seventh PME meeting, held near Jerusalem, Israel in July 1983. Three plenary papers focused on implicit models in solving problems, discussions with teachers and children, and heuristics. Contributed papers were classified under seven headings: learning theories (generalization, psychological theories, neuropsychological theories), cognitive studies in arithmetic (word problems, natural numbers, fractions and ratio, operations, applications to teaching), cognitive studies in geometry (concept formation, spatial visualization, reasoning), cognitive studies in algebra and related domains (functions, student concepts and misconceptions), computers and mathematics learning, methodology, and teachers and teaching. The supplement contains information about the conference, additional papers (including a plenary paper on research problems), and names and addresses of participants. (MNS)

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PME

Psychology of Mathematics Education

Proceedings of the seventh International Conference

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VII th CONFERENCE ISRAEL 1983
July 24 th - July 29 th

PROCEEDINGS OF THE SEVENTH INTERNATIONAL CONFERENCE FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

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PREFACE

The International Group for the Psychology of Mathematics Education (PME) was founded in 1976 at the 3rd International Congress for Mathematics Education in Karlsruhe, in order to promote international contacts and the exchange of scientific information in the psychology of mathematical education. The objective of the Group is to further a deeper and more correct understanding of the psychological aspects of teaching and learning mathematics and the implications thereof. PME Conferences were held in Utrecht, Osnabrück, Warwick, Berkeley, Grenoble and Antwerp during the six subsequent years. The 7th Conference will take place at the Shoresh guest house in the Judean Hills near Jerusalem, from July 24 to July 29, 1983.

The scientific program includes plenary lectures on topics of interest to the Group as a whole, and working group sessions with presentations followed by discussions. Plenary and contributed papers are collected in these proceedings. They have been classified under the following headings:

A. *PLENARY PAPERS*
B. *LEARNING THEORIES*
C. *COGNITIVE STUDIES IN ARITHMETIC*
D. *COGNITIVE STUDIES IN GEOMETRY*
E. *COGNITIVE STUDIES IN ALGEBRA AND RELATED DOMAINS*
F. *COMPUTERS AND MATHEMATICS LEARNING*
G. *METHODOLOGY*
H. *TEACHERS AND TEACHING*

The order in which the papers will be read at the Conference will not necessarily be identical with the order in which they appear in this volume. Complete details are given in the Conference Program.

In order to locate a particular contribution you may use the table of contents at the begining of the volume or the list of contributors at the end.

We are grateful to the contributors for their interesting papers and their cooperation. We also would like to thank the staff of the Department of Science Teaching at the Weizmann Institute of Science for the extensive help in the process of producing this volume.

The Conference Committee:

T. Dreyfus, T. Eisenberg, E. Fischbein, R. Hershkowitz,
P. Nesher, G. Vergnaud, S. Vinner.

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A. PLENARY PAPERS

ROLE OF IMPLICIT MODELS IN SOLVING ELEMENTARY ARITHMETICAL PROBLEMS

E. Fischbein
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The present research has been inspired by some previous findings refering to difficulties children encounter when faced with multiplication and division problems. Bell, Swan and Taylor (1981) have shown that children, presented with a series of problems with the same content, may change their mind with regard to the solving operation, depending on the specific numerical data. For instance, the subjects (12-15 years old) were asked about the price of 0.22 gallons of petrol if one gallon costs £ 1.20. (The subjects were asked to only indicate the operation and not to perform the computation). The common answer was: 1.20 ÷ 0.22.

Faced with the same question, but using "easy" numbers - like "£ 2 the price of a gallon and a 5 gallon can to be filled" - the subjects answered correctly: 2X5. The pupils interviewed did not consider it incompatible that the needed operation should change as the numbers changed. The authors gave the following explanation for the mistaken reaction to the first problem: the subjects considered, correctly, that the price of 0.22 gallons of petrol must be smaller than the cost of a gallon - and therefore they suggested division (Bell et al., 1981 p. 405).

A second finding related to the present investigation has been reported by Hart (1981). She mentions that subjects (12-15 years old) avoid systematically to multiply by fractions when solving a problem, though this would be the simplest way to get the solution. They prefer, instead, more complicated strategies which help them to avoid multiplication by fractions. Let me quote one example. Faced with the problem: "A 15 cm eel get 9 cm food; how much food should be given to a 25 cm eel ?" No child multiplied 9X5/3. They used instead indirect strategies, for instance: 10 is 2/3 of 15, two thirds of 9 is 6 and 25 is 15+10. Therefore one has to add 9+6=15 (Hart, 1981, p. 91).

This research has been carried out with the help of the Didactical Group of the Pisa University led by Professor Giovanni Prodi and composed of Maria Marino, Marie Nello, Maria Deri, Alessandra Betini, Paola Cerrai and Paolo Pisaneschi.

In this problem the result ought to be bigger and, therefore, the multiplication by 5/3 should have been easily suggested to the subjects, according to the interpretation of Bell, Swan and Taylor. In fact, this was not what really happened. Considering these, and other similar findings, we have set up the following hypothesis. The basic arithmetical operations remain, generally, attached to certain, implicit, greatly unconscious, intuitive models. The identification of the operation, supposed to represent the solution to a binary problem does not take place directly (problem-solving operation) but through the agency of the model. The model imposes to the retrieval process its own constraints.

Let us suppose that the concept of multiplication remains intuitively attached to the repeated addition model. For instance 3 times 5 means 5+5+5. According to such an interpretation the operator can only be a whole number. A multiplication in which the operator is 0.22 or 5/3 has no intuitive meaning (in the repeated addition interpretation). If the numerical data of the problem do not fit the constraints of the model, the retrieval process may not reach the adequate operation and the solving endeavor may be diverted or simply blocked. The subject will resort to other, indirect, ways to solve the problem, for instance to anologies, to global (sometimes incorrect) gusses, or will simply not react. This does not imply that he thinks that a multiplication by 0.22 or 5.3 has no mathematical meaning. He knows very well that 1.20X0.22 or 9X5/3 are perfectly legitimate mathematical operations. But having to solve the above "petrol" or "eel-food" problem he does not see through the problem, the solving operation. The way is blocked by the lack of correspondence between the given numerical data and the specific contraints of the underlying, tacit model. Various factors have been described which were supposed to be associated with the difficulties children encounter when facing arithmetical problems. Let me remind some of them.
a. The familiarity of the context and of the type of dimensions employed;
b. The size and type of the numbers used. A certain problem appears to be more difficult when it contains large whole numbers (hundreds) (Collis 1975); problems with decimals are more difficult than those with whole numbers (Bell, Swan and Taylor, 1981); c. The relation between the situation refered to and the solving operation. For instance: multiplicative situations containing the Cartesian product were said to be more difficult than those reducible to

repeated addition (Hart, 1980); the concordance or the disagreement between the behavioral meaning suggested by the problem and the corresponding arithmetical operation. For instance, a problem indicating reduction which is in fact solved by addition or vice versa (Nesher and Teubal, 1975; Nesher Greeno and Riley, 1982); d) Effects rigidly attached to specific operations ("multiplication makes bigger" and "division makes smaller") (Bell, Swan and Taylor 1981).

As far as we know, no attempt has been made to set up a comprehensive theory which could be able to explain these various, apparently disconnected findings. From time to time tentative, ad hoc explanations, are offered. For instance: it is technically more difficult to handle decimals, than whole numbers; verbal cues may bias the solving endeavor; a concrete context may facilitate to finding the solution; pupils remain bound to a particular meaning initially attached to an operation; most of the adolescents do not reach the formal operational stage, etc.

We suppose that the concept of intervening intuitive model may explain, in a coherent manner, most of the typical difficulties children encounter when attempting to solve a single operation problem. The main exception is represented by the familiarity of the text (terms used, situations referred to etc.) which may, by itself affect the difficulty of the problem. With regard to the nature of the tacit models, a second assumption has been made. One has supposed that the models attached to the arithmetical operations are basically of a behavioral nature. In other words: when trying to discover what is the underlying intuitive model attached tacitly by a person to a certain operation, one has to consider some practical behavior, which would be the enactive effectively performable correspondent of the respective operation. This reminds very much the theory of Piaget according to which every mental operation, including the arithmetical ones, is developmentally rooted in practical actions. But in contrast to Piaget, we assume that the enactive prototypes of the arithmetical operations may remain rigidly attached to the respective notions long after these notions have acquired a formal status. As a matter of fact, this assumption may be true not only for arithmetical operations, but for many other concepts as well.

A third assumption is that these models act to a great extent unconsciously. They manipulate the solving endeavour of a person "from behind the scenes" and, thus, their impact can hardly be controlled by the person himself. These models obey their own constraints, imposed by their behavioral nature which evidently may not fit the formal mathematical constraints of the corresponding operations. Let us summarize the above assumptions. The retrival process, which is supposed to lead from a-one-operation-problem to the solving operation itself, is mediated by a certain specific behavioral model permanently but unconsciously, attached to the respective operation.

In order to test the above group of assumptions we have to first define hypothetically such potential intuitive models and test, specifically, their effects on the solving procedures used by the subjects.

With respect to addition, one may suppose that the corresponding intuitive model is that of putting together two (or more) sets of objects in order to obtain a set composed of the elements of both collections. With respect to subtraction one may consider, a priori, at least two behavioral interpretations: a) "the take away" situation: "John has 10 marbles , he gives 4 to Jenny. How many marbles has he kept?" b) The "building up" situation: "John has 6 marbles. How many has he to add in order to get 10 marbles?" For an up to date account see Carpenter, Moser & Romberg (ed.) (1982).

Let us consider, in more detail, the operations of multiplication and division which represent the main objectives of the present investigation. One may assume that the primitive model for multiplication is repeated addition: collections, containing the same number of objects are put together. According to this interpretation 3X5 means either 3+3+3+3+3 or 5+5+5. From the standpoint of such an interpretation, multiplication is not a commutative operation. One factor represents the operator ("how many" collections or magnitudes are successivelyadded). The second factor represents the operand: the magnitudes which are repeatedly summed up. A certain number of consequencesmay be drawn from the "repeated addition" interpretation: a) the operator must be a natural number, while the operand may be represented by every kind of number or magnitude. One cannot, intuitively conceive (in the above interpretation) an operation in which a quantity g is taken 0.63 times or 3/7 times, that is g+g -- 0.63 times or: g+g -- 3/7 times. Instead, the operand may be everything. One may easily conceive the following repeated addition (even if one is not able to perform it): 3 times 0.63 = 0.63 + 0.63 + 0.63; b) A second consequence is that multiplication "makes bigger". According to the repeated addition interpretation by multiplying a quantity g (the operand) by n one gets a result which is necessarily n times bigger than g.

Our hypothesis was that repeated addition represents the primitive model generally attached to the concept of multiplication. This does not imply that the teaching of multiplication has to start necessarily this way or that the respective model is imposed necessarily by the general characteristics of the children's mind. Our hypothesis was only that, as a matter of fact, for various reasons - which have to be found - this is the primitive model which tacitly affects the meaning and the use of the operation of multiplication - even in persons with a high training in mathematics

With respect to division we have assumed two possible intuitive correspondents a) Division by partition: an object or a collection of objects are divided into a number of equal fragments or sub-collections. b) Division by quotition one tries to determine how many times a certain given quantity may be contained in another bigger quantity.

The first division model would imply the following numerical constraints: The dividend must be bigger than the divisor; the divisor (the operator) must be a whole number; the result must be smaller than the operand. The only numerical constraint imposed by the quotition model would be that the dividend should be bigger than the divisor. Clearly enough, it would depend on the structure of the problem itself which model would be activated in the given circumstances. We do not affirm that only these two intuitive interpretations may correspond to the operation of division. Certainly there are many others. Our hypothesis was that these are, in fact, the primitive models of division, i.e. the models through which division is tacitly interpreted since childhood and which continue to influence the evocation and the use of the operation of division always when an attempt is made to solve problem. It has been supposed that problems which would violate the above numerical constraints would lead to difficulties expressed in delayed reactions, in wrong solutions or even in the absence of any answer.

The above group of assumptions, imply some strange consequences (along with already known relatively trivial ones). For instance, having to solve a verbal problem in which the solution consists in multiplying 15 X 0.75, the facility to indicate the correct operation will depend on whether the decimal is the operator or the operand. Having to solve a problem in which the devisor is a decimal, the facility of solving it, will depend on whether the problem is of a partition, of a quotation or of another type. A division problem in which the divisor is for instance 1.25 is significantly more difficult than the same problmm in which the divisor is a whole number, though the division makes smaller effect does not intervene. As far as we know, no one of the already described factors may predict such curious effects.

Method

Subjects: The subjects were 243 pupils enrolled in 9 different schools in Pisa (Italy). According to grade levels there were: 98 subjects in grade 5, 102 in grade 7 and 43 in grade 9.

The Questionnaire: In order to check the above hypothesis, a questionnaire was set up, which contained 12 multiplication and 14 division problems. One has tried to avoid the interference of other factors than the numerical relationships themselves. The questions were simple, direct and refering to situations and magnitudes which were supposed to be very familiar to the subjects. In order to avoid, as far as possible, chance reactions these items were mixed with addition and substraction problems. Altogether, the questionnaire contained 42 items. The results obtained with these two categories of items have not been analyzed and consequently will not be presented. In order to avoid the effects of boredom and fatigue, the questionnaire was divided in two groups of 21 items each. In turn, each of these two groups was typed in two different ways by simply reversing the order of the questions. Consequently we got, finally, four forms. By distributing them at random in the classrooms we could also eliminate the possibility of the subjects to crib.

Procedure: The questionnaires were solved in collective sessions. The subjects were instructed to read attentively the questions before writing the answer. They were asked to indicate only the operation by which the problem may be solved without performing the calculation. An example was provided.

The Problems

The order in which the questions are presented here corresponds to that in which they appear in Tables 1 and 2. In the original presentation the various questions were mixed randomly. The arrangement in the tables is aimed to facilitate the analysis of the data.

The Questionnaire:

1. A car runs on the highway 2 km in one minute. If the speed of the car is constant, how much will the car run in 15 minutes?
2. One kilo of oranges costs 1500 lira. How much is 3 kilo?
3. From 1 quintal of wheat one obtains 0.75 quintals of flour. Which quantity of flour may be obtained from 15 quintals of wheat?
4. The volume of 1 quintal of gyps is 15 cm^3. What will be the volume of 0.75 quintals?
5. With 1 kilo of a detergent one produces 15 kilo of soap. How much soap will one produce when using 0.75 kilo of detergent?
6. One meter of suiting costs 15000 lira. How much will cost 0.75 meter?
7. The price of 1 meter of suiting is 15000 lira. What is the price of 0.65 meter?
8. For one cake one needs 1.25 hg of sugar. How much sugar does one need for 15 cakes?
9. For 1 kilo of cake one uses 15 hg of yeats. How much will one use for a 1.25 kilo cake?
10. One piece of chocolate weights 3.25 hg. What is the weight of 15 pieces?
11. A car runs 15 km on one litre of fuel. How many km will the car run on 3.25 litre of fuel?
12. On 1 litre of fuel a car runs 14 kilometers. How many km runs the car if it uses 3.70 litre of fuel?
13. With 75 pinks one makes 5 equal bunches. How many pinks are there in each bunch?

14. In 8 boxes there are 96 bottles of mineral water. How many bottles are there in each box?
15. I have spent 1500 lira for 3 etti of nuts. What is the price of 1 etto?
16. 15 friends have bought together 5 kg of biscuits. How much did each of them receive?
17. 12 friends have bought together 5 kg of biscuits. How much did each of them receive?
18. For buying one dollar one needs 1400 Italian lira. How many dollars can one buy with 35000 lira?
19. The walls of a bathroom are 280 cm high. How many rows of bricks are necessary for covering the walls if each row is 20 cm width?
20. For making 5 equal parcels one needs 3.25 m of string. What was the length of the string used for each parcel?
21. Five friends have bought together 0.75 kg of chocolate. How much will each of them get?
22. Five bottles contain 1.25 litre of beer. How much beer is there in each bottle?
23. I have spent 900 lira for buying 0.75 hg of cocoa. What is the price of 1 hg?
24. The walls of a bathroom are 3m high. How many rows of equal bricks are there necessary to cover the walls of the bathroom if the width of each row is 0.15 m?
25. In order to adorn one handkerchief one needs 1.25 m of lace. How many handkerchiefs may be adorned using 10 m of lace?
26. A Taylor has 15 m of suiting. If for one suit he needs 3.25 m, how many suits can he make from the whole piece of suiting?

The symbol hg stands for 100 gram or an "etto" in italian. This symbol is very familiar to italian pupils.

Table 1: Multiplication Problems. Categories of Answers in Percentages

o of tem	Solving operation	Grade	Categories of Answers			most frequent errors in %	
			correct	No answer	errors		
1	2X15	5	80.00	4.00	16.00	(15:2)	16.00
		7	100.00	-	-	-	
		9	100.00	-	-	-	
2	1500X3	5	95.83	-	4.17	(1500:3)	4.00
		7	89.00	-	11.00	(1500:3)	7.33
		9	95.00	-	5.00	(1500:3)	5.00
3	0.75X15	5	78.00	8.00	14.00	(15:0.75)	11.99
		7	76.60	8.13	21.27	(15:0.75)	17.00
		9	78.26	4.35	17.39	(15:0.75)	17.39
4	15X0.75	5	58.00	10.00	32.00	(15:0.75)	28.40
		7	57.45	14.89	27.66	(15:0.75) 14.89	(0.75:15) 12.77
		9	56.59	21.74	21.74	(15:0.75)	16.59
5	15X0.75	5	20.83	16.67	62.5	(15:0.75)	41.66
		7	20.00	25.45	54.55	(15:0.75) 29.0	(0.75:15) 14.54
		9	40.00	30.00	30.00	(15:0.75)	15
6	15000X0.75	5	54.00	8.00	38.00	(15000:0.75)	27.99
		7	57.45	2.13	40.42	(15000:0.75)	29.785
		9	60.87	13.04	26.09	(15000:0.75)	26.09
7	15000X0.65	5	41.67	14.58	43.75	(15000:0.65)	33.33
		7	38.18	10.91	50.91	(15000:0.65)	49.09
		9	45.00	20.00	35.00	(15000:0.65)	29.99
8	1.25X15	5	84.00	4.00	12.00	(1.25:15)	8.00
		7	93.62	-	6.38	(15:1.25) 1.12	(1.25:15) 2.12
		9	100.00	-	-	-	
9	15X1.25	5	47.92	20.83	31.25	(15:1.25)	18.75
		7	43.64	25.45	30.91	(1.25:15) 12.72	(15:1.25) 9.09
		9	50.00	40.00	10.00	(15:1.25) 5	(1.25:15) 5
10	3.25X15	5	91.67	-	8.33	(3.25:15)	6.24
		7	96.36	-	3.64	(3.25:15)	1.82
		9	95.00	5.00	-	-	-
11	15X3.25	5	80.00	8.00	12.00	(15:3.25)	6.00
		7	89.36	-	10.64	(15:3.95)	4.25

		9	95.65	4.35	-	-		-	
		5	68.75	6.25	25.00	(3.70:14)		12.50	
12	14X3.70	7	76.36	5.46	18.18	(3.70:14)	7.27	(14:3.70)	5
		9	80.00	10.00	10.00	(3.70:14)		5.00	

Results

Multiplication problems. The results obtained with multiplication problems appear in Table 1. In the first two problems both the operator and the operand are whole numbers. Practically almost all the subjects were able to solve these problems. The fact that in the second, a "bigg" number intervened (1500) did only slightly affect the percentages of correct answers.(Table 1)

The problems 3 and 4 contain the same two numbers but in 3 the operator is represented by a whole number (15) and in 4 by a decimal (0.75). At each grade level there is a difference of about 20% of correct answers between the problems 4 and 3. The grade level did not determine any significant difference.

Problem 5 contains the same two numbers, with 0.75 again as the operator. Only 20% of the subjects in grades 5 and 7 and only 40% in grade 9 solved the problem. One may assume that, in this case, the lack of familiarity of the notions ("detergent", etc.) determined this important drop of correct answers. One has to observe that while with regard to problem 4 , there is no age (grade) effect such an effect appears in problem 5. One may suppose that, with age, the notions used in this problem become more familiar (particularly the notion of "detergent"). Problem 6 presents almost the same percentages of correct answers as problem 4 though here, instead of the number 15, the number 1500 intervens as the operand. In both the operator is the same decimal 0.75 and with regard to both problems there is a drop of about 20% in comparison with the situation in which 0.75 was the operand (item 3). We have supposed that a decimal number, less familiar than 0.75 will affect still more the solving capacity. In problem 7 the operator is 0.65, and indeed one observes a drop of about 15% of correct answers compared with item 6 in which the operator was 0.75. Both problems (6 and 7) have exactly the same content! In problems 8 and 9, the same numbers intervene, but with changed roles. If only the "multiplication makes bigger" effect had an impact on the solving facility, problem 9 would not have presented any difficulty. On the other hand if only the presence of a decimal would have affected the degree of difficulty, no significant difference would have appeared between items 8 and 9. As a matter of fact it is a drop of about 50% of correct answers from question 8, in which 1.25 is the operand to question 9 in which 1.25 is the operator. There is almost no progress with age. It was even surprising that multiplication by 1.25 (problem 9) appears to be more difficult than a problem in which multiplication by 0.75 intervenes (item 4 and 6). Problems 10 and 11 also use an identical couple of numbers, this time - 15 and 3.25. It was supposed that if the whole part of the decimal will be clearly bigger than the fractional part, the decimal number will behave almost like a whole number (as if the whole part would "mask" or "absorb" the fractional part). This assumption has been strongly confirmed. At each grade level there were differences of about 45-50% of correct solutions between problem 11 (with 3.25 as the operator) and problem 9 (with 1.25 as the operator) in both 15 being the operand. The "absorption" effect (item 11) seems to increase with age (grade) and thus determining an increment of the frequencies of correct answers.

In question 12 it is the fractional part of the operator which is dominant -3.70- and thus the "absorption" effect of the fraction by the whole part should not intervene. Indeed, the results obtained with item 12 are better than those with item 11. One may assume that, this time, the subjects will "round off" the number (from 3.70 to 4) and this will help them to see the right solution (multiplication). This effect is strong enough to increase the frequencies of correct answers with about 20-30% (compared with items 12 and 9) but still weaker than the "absorption " effect (compared with items 12 and 11).

It has thus been clearly confirmed that the role of the decimal in the structure of the problem is decisive with regard to the facility to retrieve the right operation. A multiplication problem becomes difficult if the operator is a decimal (and this contradicts the repeated addition model). If the whole part of the decimal number is big enough an "absorption" effect may take place and the decimal acts, intuitively, as a natural number.

Table 2: Division Problems. Categories of Answers in Percentages

No of item	Solving operation	grade	Categories of answers Correct	 No answer	 Errors	Most frequent errors	
13	75:5	5	92	-	8	75X5	8
		7	95.75	-	4.25	75X5	4.25
		9	100	-	-	-	
14	96:8	5	79.17	-	20.83	96X8	18.74
		7	90.90	-	9.1	96X8	7.28
		9	100	-	-	-	
15	1500:3	5	66.66	4.17	29.17	1500X3	20.83
		7	90.90	1.82	7.28	1500X3	3.64
		9	100	-	-	-	
16	5:15	5	20	4	76	15:5	61.99
		7	34.04	-	65.96	15:5	65.96
		9	52.17	-	47.83	15:5	43.47
17	5:12	5	8.34	2.08	89.58	12:5	70.83
		7	38.36	-	63.64	12:5	52.73
		9	40	-	60	12:5	60
18	35000:1400 quotition	5	82	4	14	35000X1400	5.99
		7	85.11	10.64	4.95	35000X1400	2.47
		9	91.3	-	8.7	35000X1400	8.7
19	280:20 quotition	5	39.58	12.5	47.92	280X20	39.58
		7	81.82	5.45	12.73	280X20	7.27
		9	95	-	5	280X20	5
20	3.25:5	5	76	4	20	3.25X5	12
		7	76.6	4.25	19.15	3.25X5 10.63	5:3.25 8
		9	78.26	-	21.74	5:3.95	13.04
21	0.75:5	5	80	2	18	0.75X5 9.99	5:0.75 7
		7	78.72	4.26	17.02	5:0.75	17.02
		9	82.61	-	17.39	5:0.75	13.04
22	1.25:5	5	60.42	-	39.58	1.25X5 22.9	5:1.25 1
		7	74.54	-	25.46	5:1.25	18.18
		9	85	-	15	5:1.25	15
23	900:0.75	5	16.66	22.92	60.42	900X0.75	22.9
		7	29.09	40	30.91	900-0.75	3.63
		9	50	30	20	900X0.75	5

		5	20	12	68	0.15X3	59.99		
24	3:0.15	7	36.17	6.38	57.45	0.15X3	51.06		
	quotition	9	65.22	4.35	30.43	0.15X3	30.43		
		5	29.17	4.17	66.66	1.25X10	3.54	1.25:10	18.66
25	10:1.25	7	72.73	1.82	25.45	1.25X10	9.08	1.25:10	7.12
	quotition	9	85	5	10	1.25X10	10		
		5	40	4	56	3.24X15	33.99	3.25:15	17.92
26	15:3.25	7	65.96	6.38	27.66	3.25:15	12.76	3.25X15	10.51
	quotition	9	82.61	-	17.39	3.25X15		17.39	

Division

The data appear in table 2. The first three problems are of a partition type and are in accordance with the presumed constraints of the model. At grades 7 and 9 more than 90% of the subjects indicated the correct solution. It is only at grade 5 that some differences appear among the results obtained with the three items. Either the presence of a "big" number (1500) or the use of the symbol hg in the text (for 100 gr) - or both may have influenced negatively the children's decisions (question 15). (Table 2).

Problems 16 and 17 violate one of the model's rule: the dividend must be bigger than the divisor. This caused a drastic drop in the frequencies of correct answers at all three age levels. Most of the mistakes consisted in an inversion of the order of the two terms of the division (and thus leading to a intuitively acceptable operation). It is important to observe that the percentages of correct answers increase strongly with age.

The problems 18 and 19 are of a quotation type: they respect the constraints of the respective model. Despite this there is a decrement in the frequencies of the correct answers, compared with items 1 and 2. Unfortunately the questionnaire was not well enough devised in this point and one can not decide whether the drop was due to the presence of "big" numbers or to the fact that one has shifted from partition to quotition problems. Nevertheless, there is an indication that this shift has contributed by itself to the drop in the percentages of correct answers. Problem 15 (partition) also contains a "big" number but in comparison with this item, the results obtained with items 18 and 19 (quotition) were lower.

Items 20-26 contain decimals. In the first three of them 20, 21, 22 it is the operand which is represented by a decimal while the operator is a whole number. In respect to this criterion the problems do not violate the rules of the presumed (partition) model. In contrast, the rule concerning the relative magnitude of the two terms of the division is contradicted: the dividend is smaller than the divisor. Items 16 and 17 presented the same type of violation and the main tendency of the subjects was to reverse , the role of the terms. But by resorting to the same strategy for items 20, 21, 22. another difficulty would appear: by reversing the order (respectively the role of the two terms of the division) the subjects would get a situation in which the divisor would have been a decimal! It seems that, blocked by the eventuali of having to cope with the violation of another constraint (the operator becomin a decimal), most of the subjects have chosen not to reverse the order of the terms. One, then, obtains the surprising result that for items 20, 21 and 22 one gets higher frequencies of correct answers than those obtained with items 16 and 17.

In item 23 it was the operator which was a decimal and thus a constraint of the partition model has been violated. Indeed a drastic drop of the percentages of correct answers appeared. One has assumed that in quotation problems the operator may be , intuitively, a decimal provided that the dividend is bigger than the divisor. As a matter of fact this hypothesis has been confirmed only with regard to 9th grade pupils. The low percentages of correct answers obtained by 7 and especially 5 grade subjects seem to indicate that for many of them only the partition model has an effective, intuitive role. (That is to say that, if the devisor is a decimal it does not make much difference for these subjects whether the problem is of a partition or of a quotition type). It is only at grade 9 that quotition problems may be easier solved than partition problems. The difference appears clear when comparing the results obtained with items 25 and 26 (quotition problems) - more than 80% of correct answers, - with those obtained with item 23 - only 50% of correct answers. One may consequently assume that initially there is only one intutive model for division problems. One may also suppose that as an effect of instruction, pupils acquire, in addition, a second intutive model for division i.e., the quotition model.

Summary and Discussion

The basic assumption of the present research was that arithmetical operations are intuitively associated with some primitive behavioral models on the existence and influence of which the subject himself may not be aware . Such implicite models, acting, to a great extent, beyond any conscious, formal control, may sometimes facilitate the course of the solving process; but very often they may slow down, divert or even block the solving process whencontradictions emerge between these models and the solving algorithms. Specifically, in the present investigation it was assumed that: a. the primitive model of multiplication is repeated addition and b. that there are two primitive models for division - the partition and the quotation model. It was assumed that these models impose a number ofintuitive constraints with regard to the numbers used and their respective roles in the structure of the problem. If the data of the problem lead to a violation of one or more of these constraints, the subject will face difficulties when trying to indicate the solving operation. Consequently, two problems may be operationally and even textually identical and despite this their degree of difficulty may vary as a function of the types of the numbers used and their respective roles in the structure of the problem. In the repeated addition interpretation of multiplication the operator must be a whole number and the result must be bigger than the operand. In the partition interpretation of division, the divisor must also be a whole number; in addition, thedevisor must be smaller than the divident and the result must also be smaller than the dividend. In the quotition interpretation of division there is only one constraint: the divisor must be smaller than the dividend.

It has also been predicted that if in a decimal the whole part is significantly bigger than the fractional part, the whole may "absorbe" intuitively the decimal component and thus the respective decimal will act psychologically as a whole number.

In the case of multiplication all these assumptions have been confirmed. It has been found that subjects perform sensibly worse when the operator is a decimal compared with problems in which the operator is a whole number. Even if one keeps the same text and the same couple of numbers but one changes their roles in terms of operand and operator, the capacity of the subjects to solve is significantly reduced when the operator is a decimal.

It has also been confirmed that the relation between the whole and the fractional part influences the degree of difficulty. For instance in grade 9 only 50% of subjects indicated correctly the operation 15X1.25 as the solution to a problem, while 95% were able to do the same when the correct solution was 15X3.25 (in both problems, the operator was represented by a decimal). It has been found that the "multiplication makes bigger" effect, also predicted by our interpretation, acted as a very strong factor of error.

With regard to division problems it has been predicted that two basic intuitive models may be considered - partition and quotition. This assumption has been confirmed clearly only for grade 9 but not for grades 5 and 7. One may assume that, in fact, only one primitive, intuitive model develops initially with the second being elaborated with age, probably via instruction. Only at the grade 9 level the quotition model becomes a stable and influential intuitive factor. One may assume that 7 grade pupils belong to a transitional stage. They react sometimes to a quotition problem as if the partition constraints had affected negatively their decisions (items 23 and 24) and sometimes as if an already acquired quotition model facilitated their choice (item 2

An extremely interesting situation appears when considering items 20, 21 and 22 compared with items 16 and 17. In items 16 and 17 the devisor is bigger than the dividend and this fact determined a strong drop in the percentages of correct answers, compared with item 13 and 14 in which the dividend was bigger than the divisor. Most of the errors consisted, as already mentioned in reversing the role of the two numbers in terms of operator and operand. In items 20, 21, 22 the divisor is again bigger than the dividend but, in contrast to items 16 and 17 the operand is a decimal. A new conflictual, extremely interesting situation appears. According to the subjects' reactions to questions 16 and 17 one could have expected that they will tend to reverse (mistakenly) the roles of the two numbers and thus acquiring a "feasible" division But most of the subjects did not adopt in this case that mistaken strategy and in fact one gets 70-80% of correct answers at each age level (with only one exception). By reversing the order of the numbers they would have indeed obtained a "feasible" division in terms of the dividend being bigger than the divisor. But in this case, the operator would have become a decimal and this was also interdicted by the partition model. Consequently, most of the subjects did not resort to the inversion strategy and, as a next consequence they gave correct answers. The percentages of correct answers were lower compared with those obtained for items 13 and 14 (in which no model rule was violated) but much higher than those obtained with items 16 and 17. The "operator not being a decimal" rule of the partition model seems to be a very strong intuitive factor. One may assume that the "corrective" mechanism by which a division, like 0.75:5, becomes intuitively feasible (in terms of the partition model) consists in merely neglecting the decimal point (and thus seeing, for instance, 0.75 as 75).

Clearly, the processes described above are what one obtains when one trys to get an explicit representation of what happens at the intuitive, tacit level. What one gets is a chain ofconsiderations and transformations which are formally meaningless and algorithmically incorrect. But when considering the constraints of thecorresponding primitive models the whole **hypothetical** chain of transformations becomes clear and consistent.

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TELL ME WHAT YOU ARE DOING - DISCUSSIONS WITH TEACHERS AND CHILDREN

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There are few places in the world where people are satisfied with the mathematical understanding displayed by their children, whatever the criteria of success dictated by either the needs or the educational philosophy of the country. The fault may lie in the nature of mathematics (or the nature of the mathematics we choose to teach) or in the methods we use to teach it. I suggest however that unless we can somehow match the mathematics to the child very soon, we are in grave danger of losing mathematics as a school subject, accepted by educators as a necessary part of every child's education. It may be replaced by social arithmetic and the use of the calculator for the majority of children with mathematics reserved for an elite few, as the study of Greek is today in Britain. Some would consider this a sensible step but if one's philosophy of education includes the desirability of giving all children access to their cultural heritage and the products of man's rational nature then the suggestion is to be deplored. The only alternative therefore is to reconsider the way we teach the subject and how we select material of a suitable level for the pupil. The suitability depends to a large extent on the level of knowledge already possessed by the child and a discussion of the methods we might use to discover this level form the major part of this paper.

THE CHILD'S LEVEL OF KNOWLEDGE

The means by which we ascertain what children understand seem to form a hierarchy of respectability in the minds of parents, employers and educators. Test papers which are published and therefore cost money are deemed superior to those written by the class teacher. Printed matter is in its very nature thought superior to oral communication although it is this last that adults rely on in their ordinary daily life. A growing body of research is now based on data obtained from interviews although this method of assessment is less popular with teachers. There are a number of drawbacks to the interview method, as we are reminded by Carpenter, Blume, Herbert, Anick and Pimm (1982) in their Review of

Research on Addition and Subtraction

> Opper (1977) pointed out some of the procedural difficulties associated with the individual interview method. Among these were (a) the possibility that the child would not be at ease and perform naturally in the course of dialogue with the interviewer, (b) the problem of the interviewer maintaining neutrality and avoiding attempts to elicit "correct" answers, (c) the misunderstanding of language not adjusted to the child's level, (d) insufficient time for the child to reflect on the problem and to develop his/her explanations, and (e) the interviewer's interpretation of the child's actions and responses on which subsequent questions are based.
>
> One of the most serious problems with interview data is that children's explanations of how they solved a problem may not accurately reflect the processes that they actually used. The interview procedure may change how a child solves a problem, or children may have difficulty articulating the process that they really used and therefore describe another process that is easier to explain. Or they may try and second guess what they think the interviewer is looking for. Another serious problem is that the inferences drawn from an interview involve a great deal of subjective judgment on the part of the experimenter.
>
> (pg 54)

Although the list is daunting, the objections can all be made equally against the method of assessment we have been using for many years - lists of computations to be completed in a fixed time in a fairly hostile atmosphere. The greatest disadvantage of a research methodology based on the acquisition of interview data is perhaps the amount of time that needs to be spent in order to truly listen to a child and then to transcribe the interview. It is this perhaps which leads researchers to limit the type of question discussed and the number of children interviewed or indeed to try to shorten the whole process by providing 'interview' booklets which can be completed by the child with a pencil and then 'marked' by the researcher. If we limit the means by which the child can answer we limit the richness of our data. To find out what children understand we must take into account what they 'get wrong' and why they fail to function adequately in certain areas. To do this we need to provide an opportunity for the child to convey his thoughts to us.

We would hope to influence teachers with our research so we must try to design research that they will believe. This means that our interviews should be so structured that the teacher can replicate the type of discussion and so verify the results we have obtained, besides possibly adding to his own repertoire of teaching skills by using interview techniques. Teachers are engaged in working with groups of children and a class is seen as needing something in common not as a set of individual needs. The identification of a learning difficulty which is common to many is of more interest to teachers than the pinpointing of a unique situation, so anecdotal data which might relate to only one or two special cases is of less value. To influence the practice of teaching, researchers must provide information to which teachers can relate and act upon.

RESEARCH USING INTERVIEWS

The research carried out at Chelsea College over the last nine years has been focussed on classroom practice and designed to give information to teachers. We have employed interviews extensively besides collecting data in the more formal atmosphere of mathematical tests. The work of the Concepts in Secondary Mathematics and Science Project (Hart, 1981) gave us a crude picture of which aspects in eleven different mathematical topics were easy for secondary age children and which more difficult. The methodology involved the use of word problems firstly in an interview schedule and then in paper and pencil format. The latter enabled us to obtain a broad view of the mathematical performance of children in the 11-16 age range, whilst the 300 interviews enabled us to put forward tentative reasons for the different levels of difficulty.

From our testing (n = 10,000 children aged 11-16) it seemed that at least half our secondary population was restricted to the ability to solve items which required at most two steps for solution, largely involved whole numbers and could be completed nearly always by the use of the operations of addition or counting. Counting is one of the most primitive methods for solving arithmetic problems although for young children it has the advantage of rythmic naming encouraged by parents and grandparents and the satisfaction obtained from tapping with a finger. Easley (1982), impressed by the mathematical attainment in one Japanese school, suggests that one reason for the greater number dexterity exhibited by the children there was the absence of counting in Grade One.

The children in Kitamaeno School did very little counting and concentrated on partitioning and regrouping.

> With American teachers in third grade wondering, "How can I get the children to stop counting?" it was impressive to see that counting was not necessary. Counting is a terribly inefficient method but it's the foundation of our curriculum. So we wanted to see a curriculum in action where counting is not so central. Counting is one of those procedures which is very useful at times, but like any procedure, will not get you what you want, if you want mathematical thinking and greater confidence in tackling problems. As we have seen, a set of counting algorithms which children learn sets the tone, the pace, the attitude, from the first grade.
>
> (pg 23)

The fact that adolescents of 15 years were using counting, was found from asking them how they attempted to solve the problems, we would not necessarily have discovered this except through interviews.

Alternative Frameworks

There is a growing body of research in both mathematics and science education which illustrates how children are employing strategies which are not teacher taught and which were unrecognised before children were asked to explain what they were doing. One recent statement on alternative methods was made by Collis and Romberg (1981), who reported on the rules used by children aged from four to eight years (of different 'Cognitive Processing Capabilities' - CPC) when faced with addition and subtraction problems:

> Children at all CPC levels use the taught algorithm infrequently, between one-fifth and one-fourth of the number of times when it is appropriate. They appear to prefer to fall back on more "primitive" strategies such as counting which they have used successfully previously..... It is of interest to note that when the children cease to use inappropriate strategies they do not, in the main, turn to the algorithm which has been taught as the appropriate strategy. In fact, for this population, the use of the algorithm does not increase significantly with increasing CPC level.
>
> (pg 140)

The CSMS interviews tended to show that secondary school children were employing strategies in mathematics which were adequate for some of the questions they were asked to solve but unlike the algorithms they had been taught, they were not generalisable. For example, a procedure of repeated halving is adequate for dealing with ratios 3:2:, 5:2 but does not generalise to finding an enlargement in the ratio 5:3. Most secondary school mathematics is concerned with formalisation and generalisation and if the child is to succeed he must "play this game". Vergnaud (1983) states

> The formation of a concept, especially when you look at it through problem-solving behaviour, covers a long period of time, with many interactions and many decalages. One may not be able to understand what a 15-year-old does, if one does not know the primitive conceptions shaped in his mind when he was 8 or 9, or even 4 or 5, and the different steps by which these conceptions have been transformed into a mixture of definitions and interpretations. It is a fact that students try to make new situations and new concepts meaningful to themselves by applying and adapting their former conceptions.
>
> (pg 17)

To this I would add - sometimes the adaptation is a renewed allegiance to a naive version with which they feel comfortable and confident.

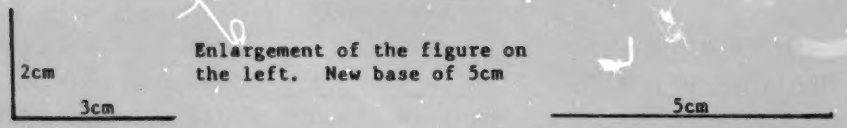
The non taught mathematical methods used by children well into their secondary schooling may be 'naive' strategies they learned when young and which are assumed by teachers and textbooks to have been replaced, or they may be invented or 'common sense' techniques. As part of the research carried out for the British government report on mathematics education (Committee of Enquiry, 1982), Fitzgerald (University of Bath, 1981) and Sewell (1981) showed that adults seldom used taught algorithms in the mathematics they used in everyday life but they cope by using their own non-standard methods

> The notation of fractions appears in some clerical and retail jobs, for instance $4\frac{3}{7}$ to represent 4 weeks and 3 days or $2\frac{5}{12}$ to represent 2 dozens and 5 singles. However, school-type manipulation is rarely found and then only in very simple cases; for instance, the calculation required to find the charge for 3 days based on a weekly rate is division by 7 followed by multiplication by 3.
>
> (pg 22)

The needs of the working man and woman of 1982 should not be the guiding principles by which we decide on the mathematics to be taught to a child who will live his life in the 21st century, so although such common sense methods prove adequate for many mathematical exercises we require children to complete, they prove inadequate if the problems are complex or involve non integers and in these circumstances often lead to error.

Errors

The CSMS data revealed that certain items on individual test papers produced the same wrong answer very often (40-50 per cent level). These errors were not restricted to particular schools or text book use and seemed worthy of further investigation. In 1980 the SSRC financed a further project at Chelsea called Strategies and Errors in Secondary Mathematics (SESM). The aims of this research were to investigate the identified errors more deeply and to try some remediation. We took errors in the topics of Ratio and Proportion, Algebra, Fractions, Measurement and Graphs and based our initial work on interviews with children who had committed the errors in which we were interested. The CSMS word problems formed the first interview schedule and we attempted to find the methods children used correctly as well as the reasons for the later incorrect answers. The underlying rationale was that consistent errors in solving a particular type of problem were indicative of a mode of thinking and not just an example of a momentary lapse in concentration. About 60 children were interviewed in each topic investigation. The methodology is illustrated by examples from the Ratio and Proportion investigation. The error being investigated in this topic was the incorrect addition strategy (Piaget and Inhelder, 1956; Karplus, Karplus, Formisanoand Paulsen, 1975) in which one enlarges a diagram by adding an amount as is shown in figure 1.



(CSMS 1976 data)	Age 13(n=800)	14(n=767)	15(n=690)
Correct answer	7.9	11.0	19.7
Answer 4	47.6	39.4	39.7
			per cent

Figure 1 The Incorrect Addition Strategy

The reasoning of children giving the answer '4' is that 3cm in the smaller diagram became 5cm in the larger by the addition of 2cm, so the new upright must also be found by adding 2cm. This same argument was put forward by those interviewed in the SESM research when the dimensions were changed so that (i) the base lines were 3cm and 10cm and (ii) the figures were triangles. The 'adders' who used this method on three/four of the hardest CSMS enlargement questions, could recognise when figures were not similar. Indeed when asked to draw the triangle resultant from the addition of 7cm to the height, their comments showed their dissatisfaction with the new shape:

"It's too long"

"It's too steep"

"Looks more tilted"

"Not as sloped"

"They're not the same"

"It's close"

The interview sample was taken from classes described as 'average' by their teachers; they had been taught some aspects of Ratio and Proportion and were not regarded as in need of remedial help. 'Adders' throughout the CSMS survey had proved to be children who could cope with a number of the CSMS items. They could for example enlarge a diagram in the ratio 2:1 and solve questions about a recipe in which the ingredients for eight people were given and those for four and six people were required (as long as fraction computation was not involved, i.e. ½ pint for 8 people. How much for 6?). The new SESM interview sample similarly gave answers of this type. In previous papers (Hart, 1981) I have referred to the correct but naive use of addition as 'building-up' to an answer. In the recipe question the ingredients for six people were found by adding the amount for four persons (the operation of halving) to the amount for two people (halving again). The 'adders' success then, was imbedded in these naive methods in which repeated addition replaced multiplication and fractions (other than '½') were avoided. It was a natural step to seek an additive method for solving the harder items in which the fraction element played a larger part. Fischbein (1983) describing some of the CSMS results comments

> And this is not only because the notion of multiplication is, intuitively, related to a magnifying effect, but also because the operation of multiplying by a fraction has no intuitive meaning at all! Multiplying 2/3 (as a magnitude) by 6 (as a non-dimensional operator) means, intuitively, 2/3 + 2/3 + 2/3 ..... What is the intuitive meaning of multiplying 6 (as a magnitude) by 2/3 (as a non-dimensional operator)?
>
> (pg 3)

INFORMATION FOR TEACHERS

SESM - Ratio and Proportion

The second part of the SESM work entailed intervention and some trials of materials that teachers could later use with children in their classes. Thus, having found a number of reasons why a child was giving the wrong answer we attempted to identify a series of constructs which matched the gaps in the reasoning of the 'adders', starting with a demonstration of the outcome of the method they were using and then stressing the operation of multiplication. The methodology was thus tailored to the error and counter to the suggestions of Gagné (1983) when he referred to rules of computation

> Hypothesis 1 is this: The effects of incorrect rules of computation, as exhibited in faulty performance, can most readily be overcome by deliberate teaching of correct rules. My interpretation of previous psychological research on "unlearning" is that it is a matter of extinction. This means that teachers would best ignore the incorrect performances and set about as directly as possible teaching the rules for correct ones. An unpreferable alternative is to make students fully aware of the nature of their incorrect rules before going on to teach the correct ones. It seems to me this is very likely a waste of time.
>
> (pg 15)

In Ratio and Proportion I sought to distinguish between a schema and concepts and the skills needed for a successful demonstration of them within school mathematics. I took as a matter of educational belief that there were children within the group of 'adders' who were 'ready' to move their level of understanding if they were only given the right information at the time appropriate for their needs. An additional condition was that ideas for remediation should be in a form that teachers could and would use.

The module for Ratio and Proportion addressed itself to four areas in which the adders appeared to be deficient:

i) the recognition of the inaccuracy of the diagram resulting from the use of the incorrect addition strategy

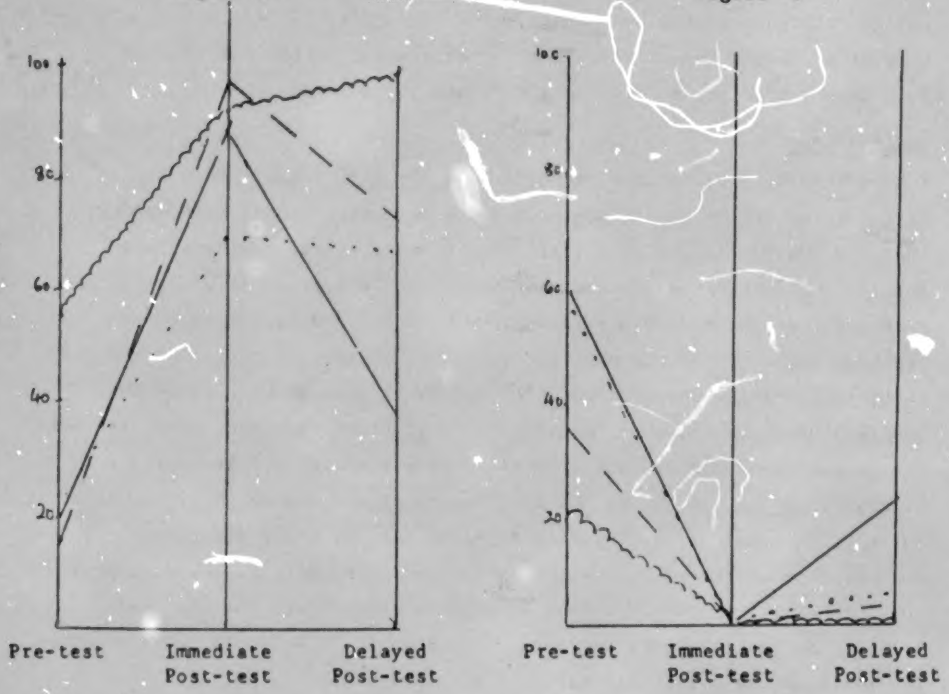
ii) the need to see that the crucial arithmetic operation involved was multiplication

iii) the possession of a skill which would enable the child to multiply decimals (or fractions). This is distinguished as a separate entity from the understanding of what is needed to procure an enlargement

iv) the possession of a method which would enable one to find a scale factor, given a dimension and its enlargement.

The module was tried with small groups of adders, then half classes and finally by teachers with classes of children whom they considered to be of 'average' ability. The results for the four teachers who used the materials for two weeks teaching are shown in figure 2. In figure 2b we can see that the incorrect addition strategy has completely disappeared at the immediate post test but some children have reverted to it by the time of the delayed post test, 11 weeks later. Of the 24 children in the four schools who would have been designated 'adders' on the CSMS test because they used the incorrect addition strategy on three/four of the four hardest questions, 23 were no longer in this category on the delayed post test. Solving the items correctly is more difficult and as can be seen from figure 2a, although the performance of every class improved, school four's results show a sharp decline between immediate and delayed post tests.

Figure 2a　　　　Figure 2b



Correct Answers　　　　Addition Strategy Answers

.... School 1

---- School 3

—— School 4

〰〰 School 6

The percentage is obtained: $100 \times \frac{\text{Number of answers given}}{\text{Number of answers possible}}$

Figure 2 Results for the school trial

Positive intervention by the teacher, designed to meet a specific and identified need has been shown to be effective. It would seem very often that intervention which is too general or too far removed from the misconception fails to effect any improvement.

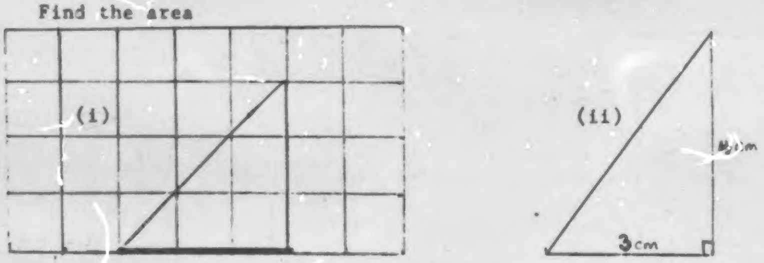
Implications

I have been told by various teachers that the CSMS results mean that half our population is incapable of doing secondary school mathematics; that the subject is for an elite; that primary school teachers have failed or that secondary school teachers have failed; that practical work and concrete aids/manipulatives need to be used to a much later age than is currently the practice etc. All of these are of course matters of opinion and even politics and their generality hides the subtlety of the problem of teaching mathematics to children. The statement "he is not ready because he is at the concrete stage" can be as crude a diagnosis in its own way as "he is 14 years of age and so should be able to do ....". Neither necessarily describe the child and its needs. Our evidence of the existence of child-methods and alternative frameworks suggests a picture of a far from smooth transition from the state of reliance on concrete aids to 'formal' mathematics. We would agree with the statement by Driver and Easley (1978)

> It has been suggested that in designing instructional sequences attention needs to be given not only to the sequence of operations or ideas to be taught but to the operations or ideas pupils tend to apply spontaneously to tasks.
>
> '...until one understands what students do spontaneously one will not be able to demonstrate the limits of this approach to them' (Case 1976).
>
> Knowledge of pupils' alternate frameworks has been used in constructing learning tasks in science.
>
> (pg 78)

The path is littered with the jagged rocks of methods found reliable by the child when he was young and in which he still places his confidence, the methods he has invented and which cause him to be talking on a different level to his teachers and finally, the belief that it is all magic anyway. The teacher must find out the location of these rocks and without destroying them lead the child to a higher plane. The division between the higher plane or 'formal' mathematics and the level of the child may, to continue the allegory be rather a large plateau, a huge chasm or a small step, we have very little information on this. We do for example know from our CSMS data that the two questions in figure 3, given to the same children on the same day, gave very different results.

There seems to be a large gap between the difficulty of question (i) which can be solved by counting and adding and the formalisation needed for question (ii).



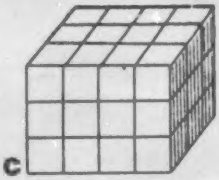
Age	12	13	14 yrs
(i) percent correct	77.5	86.9	91.4
(ii) percent correct	31.4	38.7	47.5

Figure 3 CSMS area questions

The Path to Formalisation

Teachers in Britain who follow the 'official line' in teaching would approach a formal mathematical statement through a wealth of practical experience for the children but there would come a time when they expected the class to deal with symbols and abstraction without the concrete aids. At Chelsea we are currently looking at the transition stage between practical experience and formal mathematics for children aged 8-13. The step from one to the other may be too great and experiences which provide a bridge between them may be called for. I give as an anecdotal illustration a conversation with a ten year old, Paul, who jas just spent a month doing practical work relating to volume prior to learning that the volume of a cuboid is given by "V = lxbxh". The problem used is the well known Piagetian example of decomposition of a cuboid as shown in figure 4.

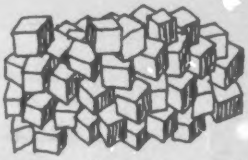
Block 'C' is made by putting some small cubes together:



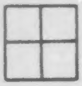
C

How many cubes make this block 'C' if there are no gaps inside? ....

c) All the cubes from block 'C' are put in a pile:



I am now going to use all these cubes from block 'C' to build a "sky-scraper" so that the bottom floor is 4 cubes



How many cubes high would this "sky-scraper" be from the ground? ....

Figure 4 Volume questions used on interview

(Interviewer: I - Paul: P)

I: I have a picture here. That is made like that (showing cuboid) - that is just the picture. Then I tear it all to pieces, to get these bricks like that. Then I build it up again. This time I build it up on top of that (pointing to ⊞) How high would it go? (Repeats briefly).

P: Can I do it any way, but just find the answer?
I: I would prefer you not to break it up but if you need to break it up, well go ahead. Can you do it without breaking it up?
P: It's 3 fours in each layer
I: Good
P: So, 1, 2, ... that makes 4 there. Another .. (long pause)
I: Can I tell you what you said when you worked that out? You said three fours. Then you wrote that

3 4
3 4
3 4

I: Then you said three fours and you wrote that. What are you going to do?
P: Just ... it's 12 isn't it?
I: Twelve fours?
P: Twelve into ...
I: Twelve is what ...? Is it all the lot. Or is it a layer? Or is it three fours? Or what?
P: It's ... kind of how many fours is three times 12 fours.
I: Could you show me 12 fours, on that, you see that block there. Where are the fours.
P: Do you want me to break it up?
I: No, you didn't break it up to get to that answer. Did you?
P: There's a four here. You said 3 up, to make it how high.
I: Are you giving me the answer then, 12.
P: Yes
I: So, is it 12 high?
P: Yes
I: I don't understand. That is the drawback. Do you think you could explain to me very slowly, what you were doing.
P: Alright.
I: I found that bit I thought you said three fours and that was a layer, was it not?
P: This is four, four and four, so it's three. Three ...
I: I see, three little bits of four
P: Yes... three time it - sounds a little bit like a table
I: That is what I thought you were doing, you see. But you are not. There are 3 fours in that top layer, I agree.
P: Then there's another four and another four. That's what I've written here and that all comes up to 12.
I: Twelve fours?
P: Yes.
I: So I see.
P: And that's the height.
I: Splendid. Now I understand. So that is the height of our skyscraper?
P: Yes... Do you understand now?
I: Yes. I'm not sure I agree but I understand.

Paul has not moved from counting cubes to multiplication but to a collection of fours (⊞), he is not at a formalisation stage but not completely tied to the concrete either.

The research design of 'Children's Mathematical Frameworks' involves the teacher in writing a scheme of work and justifying it to us. Then we interview six children in the class just prior to the lesson(s) involving the formalisation, to try to ascertain the nature of the pre-requisite knowledge (identified in the scheme of work) that they possess. We listen to and tape record the lesson(s) when the teacher formalises the idea and then we talk to the six children again, immediately and after three months have elapsed in order to see whether they utilise the formalisation and then how the nature of their understanding has changed in the intervening period. From this we hope to obtain evidence which sheds light on the source of a number of misconceptions and child methods shown by secondary school children.

CHILDREN AND TEACHERS

The case has been made that interviews with children enable us to discover considerably more about the nature of the child's understanding than we could from his performance on a pencil and paper test. Let us extend this idea to include other types of verbal interchange.

Discussion and Debate

Adults place particular regard on oral communication; we all attend a conference because although we can read the papers we value the opportunity to hear people and to ask them questions. We seem not to think children might benefit from the same type of interchange. I suggest that children are expected to ask only the questions the teacher has planned for and to which the answer is known.

We tend to think that we are protecting children when we omit the awkward example or slide over the less than satisfactory explanation. It is thought children are less confused if many of the complexities of a situation are disguised. For example in lxbxh, is l sometimes b and sometimes h or is the label permanently attached to a side? We essentially leave the child to discover these inconsistencies when he is alone. We tend to think that children's opinions are not worth heeding and although we may sometimes listen to a child explanation, we quickly interject with out interpretation of the situation. I suggest in fact we should make a positive and penetrating effort to find out what the child is doing, as Bauersfeld (1978) said

> Erlwanger's case studies are related to programs from individual Prescribed Instruction. His documentation of students' mathematical misconceptions and deficiencies demonstrate how mathematics learning can be damaged by restricted teacher-student communication - a restriction which leads to the nearly total absence of negotiations over meanings.
>
> (pg 5)
>
> Teaching and learning mathematics is realised through human interaction. It is a kind of mutual influencing, an interdependance of the actions of both teacher and student on many levels. It is not a unilateral sender-receiver relation. Inevitably the student's initial meeting with mathematics is mediated through parents, playmates, teachers. The student's reconstruction of mathematical meaning is a construction via social negotiation about what is meant and about which performance of meaning gets the teacher's (or the peer's) sanction.
>
> (pg 19)

The effectiveness of teachers is closely linked to amount of developmental work (as opposed to practice) they are prepared to introduce into their lessons, Grouws (1982).

> Development is a part of most mathematics lessons in which the teacher actively interacts with pupils. In very general terms, development can be thought of as that part of the lesson devoted to the meaningful acquisition of mathematical ideas, in contrast to other parts of the lesson such as practice or review.
>
> (pg 5)
>
> If development is viewed in a global way, then teachers' attention to the meaning of material presented is vitally important, and development must be understood in terms of promoting student thought. Future classroom research must study more extensively the content that is presented to students rather than instructional time per se.
>
> (pg 17)

Both of these aspects can be achieved, I suggest, if one involves the children in a discussion of the topic under consideration. Easley (1982) after his research in Japan has been trying to encourage some Illinois teachers to build-in debate and discussion as part of their normal teaching.

> In Kitamaeno School, we had seen teachers urging students to find as many methods as possible, and urging students to learn several methods for each kind of problem. This approach contrasts with that of the teachers we were working with, who would feel happy if they could teach one method well for each kind of problem.
> (pg 137)

Some further research carried out by Malvern and Bentley (1982) adds weight to the argument that verbal communication between teacher and child could be a rewarding experience for both.

> The aim of the Project was to augment the routine class teaching of six to eight year olds with some extra work in mathematics, increasing the scope for discussion, and paying particular attention to the use of appropriate, simple, unambiguous language. The purpose of the evaluation was to ascertain if any measureable improvement in mathematics was achieved. To provide the extra teaching six primary teachers were seconded to the Project for two years. All six were experienced teachers but none was particularly specialised in mathematics before the Project began.
> (pg 1)

> On average an extra six hundred minutes mathematics teaching was provided by the Project teachers to each infant pupil and six hundred and twenty-five minutes to each junior pupil. Again on average this means each pupil received 17 or 18 minutes per week extra mathematics taught in a small group of between 3 and 5 pupils for the most part. Some pupils received much more, some much less but the bulk of the pupils received about the average. Fifteen to twenty minutes a week is a modest amount of time compared to the normal class time given over to mathematics teaching estimated at between 5 and 6 hours per week.
> (pg 2)

> We can say without equivocation that the measured achievement in mathematics was enhanced by the Project. The enhancement was large in scale, and the results overall showed a major improvement had been achieved through this work.
> (pg 2)

The enhanced performance of the pupils is accounted for by a number of features of the project, the authors mention four, one of which is:

> Small Groups and Discussion. The Project teachers worked with small groups. This allowed not only a lot more individual attention to be given to each child, but also the encouragement of discussion of mathematics. It was the Project teachers' intention to promote talk about the work among the children

> both to improve linguistic skills generally and to make mathematics active and lively. The small size of the groups was important in creating the circumstances where each pupil was able to contribute, as well as resulting in the teachers' guidance being quickly available immediately it was required.
>
> (pg 3)

CONCLUSION

In this paper I have attempted to provide an argument for research which is based on interviews by illustrating the advantages with results and insights obtained in the CSMS and SESM research at Chelsea. Children often use methods for solving mathematical problems that are far removed from the algorithms they are assumed to be utilising. The process of teaching children is very complex and we do a disservice to both teachers and children if we pretend it is straightforward. The subtlety of approach needed might best be accomplished by greater verbal interchange in the classroom, including discussion and debate. We must surely learn to change so that the future is not as David Page (1983) sadly reflected on the introduction of 'New Math' into American schools:

> Teachers treated the new ideas in mathematics as they had learned to treat other educational innovations: put in the required time at the workshops and "wear wide ties while they are in style". Then go back to your classroom, close the door, and do what you have been doing.
>
> (pg 1)

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IS HEURISTICS A SINGULAR OR A PLURAL?

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Your first reaction to this question might be "a plural" because of the ending "s" but this would be too rash a reaction. Indeed, "Mathematics" is a singular in spite the terminal "s", — a virus like particle that has meanwhile been incorporated into "physics", "economics" and quite a few other nouns.

In order to answer the question in the title I first consulted my own English dictionaries, but I could not even locate the adjective "heuristic" between "hetman" and "hew", and only in a Dutch - English dictionary could I find this adjective as a translation of its Dutch counterpart. So I went to our University Library. Neither the adjective nor the noun "heuristic" were recorded in any English encyclopedia whereas its Dutch, French and German counterparts occured as early as the oldest encyclopedias I consulted. With the dictionaries I fared a little bit better but I had to resort to the biggest to encounter at least the adjective "heuristic".

What regards the noun, the older ones knew "heuretic", whereas either heuristic or heuristics as a noun did not occur until the sixties or seventies of the present century. Needless to say that in the continental vocabularies the corresponding adjectives and nouns have been terms of good old standing. Of course the lack of terms does not necessarily imply that of the concepts covered by those terms. Perhaps you would not believe it but it is a fact that about a quarter of a century ago the geometrical term "congruent" did not yet exist in standard English though it may be taken for granted that British teachers and students knew as much about congruent triangles as did their continental peers.

In spite of dictionaries and encyclopedias the adjectives and nouns "heuristic" and "heuristics" have now acquired civil rights in the English educational — or at least mathematical educational —literature. But you will understand that the question I asked in the title is not a terminological one. For a French title I would not have chosen a question but the title would have simply been "L'heuristique et les heuristiques", in German "Heuristik and Heuristiken", but for reasons of uncertainty about the vocabulary I could not do so in English, which explains the strange title. But even without this explanation you would have understood that the query in the title is not a linguistic one but has a more profound meaning.

I should confess that "heuristic" and "heuristics" do not belong to my own educational vocabulary. This might be just the reason why I pay attention to these nowadays quite frequent terms.

Everybody in this audience will remember the story about Archimedes, told by Vitruvius in De Architectura:

> Joyfully he jumped out of the bathtub and naked he ran home,
> announcing loud that he had found what he had searched for.
> Indeed while running he exclaimed: Eureka! Eureka!

"Eureka" means "I found it", but even with Vitruvius' story one can doubt what Archimedes had searched for and found. Vitruvius tells that Hiero,the king of Syracuse, had asked Archimedes to find out whether a golden wreath he had ordered was pure or whether the goldsmith had cheated. Unfortunately it was not a great discovery like the famous Archimedean principle - as quite a few people believe - that led Archimedes to act as he did, but it seems to have been the simple observation that the deeper he stepped into the bath the more water flew over the edge. Indeed in order to compare the specific weights of pure gold and of the wreath metal, it suffices to compare the volumes of displaced water. Scientifically this might be a minor observation but as I observed with a little boy, psychologically it can be a breathtaking discovery.

I started with Atchimedes because he was also the first mathematician, and for a long time has remained the only one in history,who granted us a look into his mathematical kitchen; who told us about a series of theorems not only how to prove them but also how he found out them. This he did in his "Ephodos" (the approach) — a palimpsest discovered as late as 1906. In the introduction addressed to Eratosthenes he wrote:

> For some things, which first became clear to me by the mechanical method, were afterwards proved geometrically, because their investigation by the said method does not furnish an actual demonstration. For it is easier to supply the proof when we have previously acquired by the method some knowledge of the questions than it is to find it without any previous knowledge. *

Archimedes' Ephodos shows that he discovered theorems that we would now prove by integration, by the method of indivisibles, whereas the official proofs he published were fashioned according to Eudoxos' epsilontics like method.

How has the whole of mathematics we can nowadays boast of been invented? We know a little bit about it from history. That is to say, history tells us how people simplified proofs devised by their predecessors, how they improved or generalised old concepts, how they polished unwieldy definitions, how they put

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* Translation: E.J.Dijksterhuis, Archimedes,1956.

whole chapters of mathematics upside down, how they transformed ideas published earlier. But what do we know about what happened in the mind of the individual mathematician who invented something or improved an invention? Even the inventor couldn't possibly tell us — Van der Waerden's story about Baudet's theorem is a noteworthy exception. All we can do is advancing conjectures. The inventor left us a clean copy after destroying the scribblings and the rough draft. This, indeed, is how problems are approached: heuristically, by searching, at random and intentionally, by finding, by serendipity or systematically. This, then, is heuristics: the scribblings, as opposed to the clean copy as it is printed.

× × ×

Since we know nothing about how Euclid (or one of his predecessors) arrived at the proof that there is an infinity of prime numbers, let us advance a conjecture. Arrived at the proof? No, at the idea of this infinity, which I would guess was preceded by a more original one: drawing up a list of all prime numbers.

You know the sieve of Eratosthenes, a way to sieve out the prime numbers. Let us think the natural numbers in a row, starting with 2 (since 1 is trivial). 2 is prime, the only even prime number; thus all other even numbers can be struck out. What is left, is the odd numbers, thus of the form $2n+1$. The first among them, 3, is again a prime number, and all its true multiples can be struck out. What is left? The numbers of the form $2.3.n \pm 1$. The first of them is 5. All its true multiples are struck. What is now left? The numbers of the form $2.3.5.n \pm 1,7,11,13$ (with after the ±-sign "relative" primes with respect to 30).

Continuing this way one discovers how for any set of prime numbers $p_1,\dots,p_n$ a new example can be found: multiplying them and adding 1, one gets a number that is not divisible by any old prime; thus it is a new one or at least divisible by a new one. But to get a new prime one could as well have formed from the different primes $p_1,\dots,p_m$ the expression $p_1\cdots p_k + p_{k+1}\cdots p_m$.

Is this the way they did it? I do not know. But if I am right it shows the underlying heuristics, the scribblings they threw away in order to dumbfound the learner by the definitive proof of "$p_1\dots p_n + 1$, and so on".

In instruction "heuristic" is the opposite of "apodictic". Not: take it or leave it. But: search and find it. Yet to tell the truth: Besides the learner there is a teacher. Rather than throwing the learner into a pool, and saying "swim", he considers it as his duty to aid his pupil: by means of a query he utters, a situation in which he puts him, by a slight push, by a hint, and if he knows how to organise it, there might be a crowd of pupils who push each other (or cross each other).

x x x

It is what they call a thought experiment. In his mind the teacher has prepared a clean copy, which he expects the learner to produce, and even a somewhat vague series of scribblings leading to the clean copy — a plan, which in the actual experiment must be modified according to the student's cooperation. This is what from olden times they called heuristic instruction, quite unlike the modern "problem solving", which can mean anything from letting the pupil muddle up to having him tied to leading-strings.

Polya, in a number of books, has illustrated heuristic learning by marvelous examples but he was also well aware of the fact that there was no Polya standing behind or at the side of each of his readers in order to intervene at the just moment in his learning process. An author who knows the right way (and that is what Polya is) lets his readers take a part in his thought but he does not take a part in his readers' thoughts — indeed, this is impossible. The only thing he can do — and Polya did it marvelously — is to assist the mathematical problem solver with advices, a collection of advices from which he can make his choice in order to tackle the problem — advices such as: consider a special case, try to generalise the problem, suppose you have solved it, use a drawing, look for a similar problem, and so on.

This kind of advices is now called heuristics, meant as a plural, whereas Polya as far as I can oversee, if he uses the word "heuristic" (singular) means that process of searching which I have tried to characterise. What they call nowadays heuristics (plural), I had the habit to call strategies and tactics — concepts well distinguished in military science: strategy that is concocted at headquarters, and tactics that is exercised on the theater of war.

In mathematical activities the borderline between both of them is more fluent and its course can depend on the circumstances. For the student who starts with algebraic word problems the advice "call the unknown thing x" is a new strategy, but as soon as he masters this trick it is degraded to a tactic of more or less automatic action.

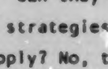
Our mathematical instruction is designed for any kind of problem to teach pupils a system of fitting tactics. These can be quite a lot, for each particular test item, say, four or five. This kind of instruction functions more or less in algorithmic arithmetic, that is, sometimes more and sometimes less. But as long as one stays with algorithmics it means that the pupil is told what arithmetical operation to carry out. If confronted with word problems pupils often find out their own tactics, which may, or may not, be given the judge's blessing. In algebra textbook authors and teachers try to provide the learner with a lot of tactics, which unfortunately do not function as automatically as those for algorithmic arithmetic. How many wrong ways there are to apply these tactics — I need not exemplify it. The literature about it is enormous and amusing, if it is not distressing — tactics that do not function because they are not incorporated into strategies.

But what about strategies? Can they be learned? Of course they can. But - another question - can they be taught? It depends on what one means by teaching. A list of strategies to be consulted in a problem situation — crossing out what does not apply? No, that is of course not what I mean. My idea of teaching strategies is making the one who solved a problem, afterwards conscious of the strategy he had followed — so sharply that he stores it in his memory and still vaguely enough to prevent him from sticking to it. Indeed, strategies are only valuable if they allow for flexibility of application. Rather than apodictically they should be acquired heuristically. And this brings me back to heuristic learning according to my terminology. To the problem of heuristic learning — I should say while at the same time confessing that I did not solve it.

x x x

What is the problem? Let me illustrate it as concretely as I can. After a straining walk I am sitting with a 12 years old boy in a train but his mind is still fresh enough to allow me to shorten the time with a bit of mathematics. I draw a triangle and ask him how much its angles could be together.

(fig.1)



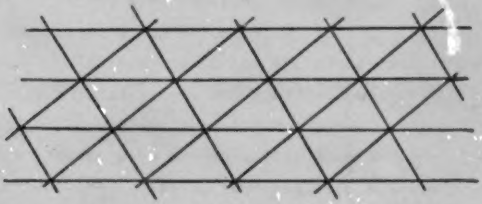
There is no protractor around but apparently he does not feel it as a lack. Some time in the past, probably not so long ago, he has grasped that such a question in such a situation does not require other means than thinking. I like to stress this because this is an attitude that even after many years of mathematics instruction quite a few adults have not yet acquired: interpreting a situation appropriately — in the present case the situation of being seated in a train, with no protractor around and nevertheless asked a question about angles — a human on top of a mathematical situation.

Perhaps my query was too sharply focused. Maybe I should have asked him: "Would there be some remarkable thing to be told about the sum of the angles?" Or would I not better have drawn different triangles and asked about that with the biggest sum of angles? Or having him compare that of a triangle and a quadrilateral?

Anyway he could not answer my question. So I tried a nudge. Imagine we would cut out the triangle - of course we had no scissors - and make more of this kind. Could we pave the paper with them?

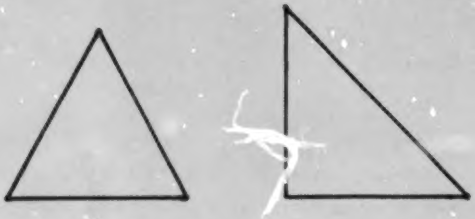
After a first failure he produced something like.

(fig.2)



Honestly I must add that he had encountered such a pattern earlier in another context. I ask him a few questions on uninterrupted straight lines in the pattern and how many kinds there might be — questions that with a ruler at hand I would have asked differently. I continued with asking questions about equal angles, and no wonder he had soon found out that the sum of angles in any triangle was 180°. Of course we also tackled the equilateral and the isosceles right triangle.

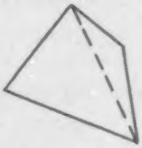
(fig.3)



Then it was the turn of the quadrilateral, in which he immediately drew the diagonal required to get the sum of angles 360°.

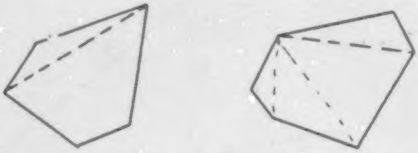
A side-leap to the square and the rectangle, and then the pentagon.

(fig.4)



Here something went wrong, that is,not according to my thought experiment. He drew one diagonal only, splitting the pentagon into a quadrilateral and a triangle, and so he kept it with the hexagon: one diagonal splitting it into a pentagon and a triangle. Strangely enough I had not expected it and

(fig.5)



so I was ill prepared. How could I now decently pass to the 1000-gon when he knew nothing about the 999-gon? My reaction was awkard as it often is if my thought experiment is falsified. I replaced his inductive idea, which looked clumsy, with my own , and you can imagine how this worked:

in the 3-gon the sum is 1 time 180°,
in the 4-gon the sum is 2 times 180°,
in the 5-gon the sum is 3 times 180°,
in the 6-gon the sum is 4 times 180°

(with heavy stresses on the numbers 3,1; 4,2; 5,3; 6,4), thus for instance

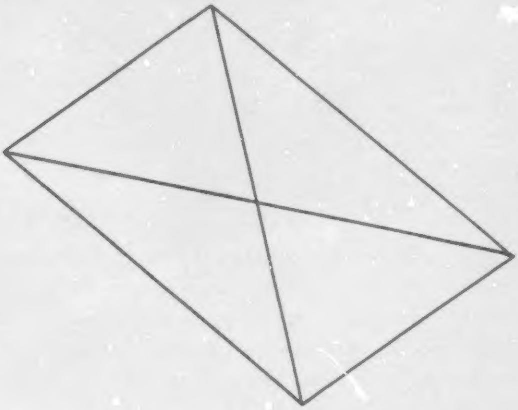
in the 1000-gon ...?

No wonder he grasped it. It was one of my worst performances. I was even so insolent to ask him about the n-gon — a question he was right not to understand since he had not yet the linguistic tool of $(n-2\ ).\ 360^\circ$ at his disposal

There are more ways to deal with the sum of the angles in a triangle: Starting with a rectangle where the sum of angles is obvious. (Both the existence of rectangles and the posibility of paving the plane with congruent triangles are equivalent with Euclid's postulate on parallel lines.) Then passing from the rectangle by means of a diagonal to the right triangle. Or shouldn't one start with the right triangle and hope that the pupil will complete it to a rectangle? It would be one more opportunity to let him act heuristically. If he fails, one can still switch to the rectangle in order to return to the triangle. Then after the right triangle the general one, hoping to see it split into two right ones. But I am afraid to achieve this, a kick rather than a little push is required.

But let us continue with the lesson in the train. I drew a parallelogram with its diagonals. Do you see things that are equal? He first saw unequal things only, which in fact are more striking. With little pushes "another pair" he arrived at stating that the diagonals halve each other.

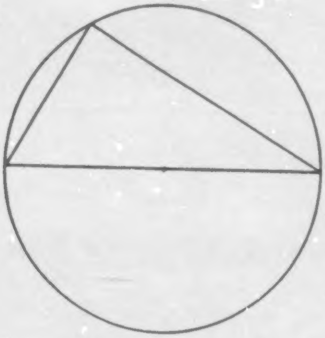
(fig.6)



How can you see it? I had to appeal to ideas he had already learned at school: "congruence", and "how many times does a certain figure fit into its own hole?" He succeeded. In his mind he turned the parallelogram around. What does happen with the diagonals and their intersection? Answering this question means solving the problem. Then spontaneonsly he motivated in the same way why the isosceles triangle has equal angles at the basis.

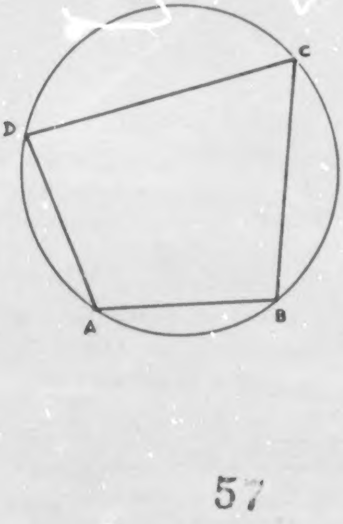
There were circles printed on the sheet of paper we worked on. I inscribed a triangle upon a horizontal diameter. What do you think about that angle at the top? Of course he recognised it as a right angle. "Are you sure that it is exactly a right angle?" "Yes", and I was surprised about the motivation. He completed the triangle to an inscribed rectangle, and for that matter, an "oblique" one. (He knew before that a rectangle is characterised among the quadrilaterals by the equality of the diagonal pieces.) It was an instructive experience. In order to pave the road more smoothly, I could have started with the inscribed rectangle, and with a diagonal added, obtained the triangle upon the diameter. It would have been safer but I would have had a heuristic opportunity let slip.

(fig.7)

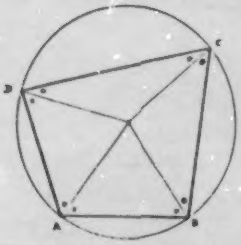


I had the courage to go even further. In one of the printed circles I inscribed a quadrilateral. To be safe I drew it with the centre M of the circle inside.

(fig.8)



(fig.9)



I let him M connect to all vertices and indicate all pairs of equal angles. What do you think about the angles A and C? He saw immediately that they were 180° together. And what about B and D? The same. My next step was D replaced with D' on the same arc AC. Almost spontaneously he drew from $\sphericalangle B + \sphericalangle D = 180^\circ$ and $\sphericalangle B + \sphericalangle D' = 180^\circ$ the conclusion that the angles D and D' on the same arc were equal.

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With this story I did not solve the problem of heuristic instruction nor did I make it easier. Maybe I even complicated it, and this was my intention, indeed: to go to the heart of the matter. One pupil per teacher — it can be a privilege (for the pupil and the teacher). I do not mean: private lessons, which are too expensive to have precious minutes wasted by aimless search.

A more realistic setting is heuristic instruction on the blackboard in the classroom. According to the number of pupils the probability to succeed is increasing. It would be strange if not at least one of the pupils went along with your plans and allowed you to transform your thought experiment into a real one.

It would be a variant to have the pupils work individually and after the question who succeeded to have the - imaginedly - fortunate ones demonstrate their solution on the blackboard or the overhead projector.

One can also have the pupils divided into groups, parcel out one's attention as a teacher, and step in to help a particular group at the point where one feels they have arrived.

On the other hand one can organise the realisation of the thought experiment beforehand by means of a sequence of worksheets that should lead the workers to the goal. This sequence can be organised rigidly, with little steps, or more loosely, with the proviso of white spots where one would intervene if the learning process stagnated.

Finally one can charge a computer with the thought experiment, strictly programmed, or with intermediate references of the pupil to the teacher.

How should it be done? I cannot tell it as I cannot tell how to compare methods to find out the best. I have seen reports of class discussions where the teacher refrained from intervening and restricted himself to watch the discussion and where the result was such that when reading them, one pitied the pupils and regretted the waste of time. And I know programmed instruction that the learner has to undergo with eye-flaps. Heuristic instruction as I see it, is an art of meting out the right dose of constraint and freedom.

Anyway heuristic instruction is quite another thing than the fashion of instruction of heuristics (plural).

x x x

I am not yet finished. When I was invited to deliver an address, my first choice of a title was "PME?", that is, "psychology of mathematics education" followed by an interrogation mark. It soon appeared that the design I had in view, lacked the ring of concreteness I prefer. However, I did not abandon my original idea; I rather delayed its realisation. As much as concreteness I like analysis of composite concepts. PME is such a complex. Let us start at the end. "Education" has a lot of meanings. In 1978 P.Suppes published a book "Impact of research on education", consisting of nine "case studies". However, education in the sense of an activity in the classroom is almost absent in this volume. Education as understood by its authors is educational research, and the impact as felt in it is one of less educational on more educational research.

Education as understood by myself is about learning and teaching as processes, taking place in a more or less organised way. I agree it is not the general view. When scrutinising a few volumes of some research journal on mathematical education, I could not find anything that matches my definition. Most of the papers were concerned with states rather than processes, instantaneous photographs of products of previous education or non-education. To be sure there were papers among them where two states were compared with each others, in the way they did it in old-fashioned advertisings: "before the treatment and after". What happened in between was indeed a treatment as they called it rather than a teaching-learning process. There were catalogues of errors of learners but nothing about the learning and unlearning of errors.

I also scrutinised a few volumes of our PME proceedings, and I am happy to tell you that about a third of the contributions were concerned with what I like to call education, that is learning and teaching as a process.

After the E of PME let us turn to the M. I will be short about it. Not because of unconcern but rather because I would not know what to add to the old controversy about the mathematics that is being taught and the one that should be taught. Anyway personally I favour innovation above conservation and I would like PME to share this preference.

Finally the P of PME. I do not like definitions but I think what people agree on is that psychology is about behaviour, say, of humans (though I do not exclude higher animals). Their methodology, however, covers a spectrum as broad at that of light, from the cool specialist of measurement to the hot visionary. I think PME is working in a temperate climate. But even a temperate climate knows temperature gradients.

Psychology is about behaviour — all right. But some time ago they enriched "behaviour" with a plural "behaviours", which in its most extreme version means knacks and tricks. Studying behaviours can be useful provided one does not forget about the behaviour behind the behaviours.

You understand how this analysis of the P in PME is related to the title of my address. It is again the choice between a singular and a plural, and my own is the singular. Thereis a trend in psychology, as there is in other social sciences to come to grips with global ideas by subdividing, by grinding, finally by atomising them. It is not my view on psychology. My view on global ideas is the paradigm. My search is for paradigms of mathematics development and education, appropriate to better understand learning and teaching. This, then, was the kernel of my exposition: paradigms of right or wrong behaviour of learners and teachers.

Where to look for them? In learning processes, of course. Short term ones is the easy case, the fingering exercise. Long term ones are found in curriculum development if understood as educational development — a promising territory.

How to find them? By focusing on and singling out what looks important. How do I know what is important? By PME — now with an exclamation mark.

When rereading the last few passages, which were to mark the end of my address, I felt sorry. Sorry about the wrong I might have done to people that share alternative ideas about what means PME, among which — strange bedfellow — my other self. Sorry about having atomised PME into its components — an activity I dislike if performed by others. But there is no focusing without blurring. If this is the cause of my wrongdoing, I could not avoid it.

I am going to resume my analysis. "Educational Psychology" is of old standing. To the "P" and "E" we added the "M". Is it a restrictive or replenishing adjective? It is the wrong question. Let us think once more about it. "Educational Psychology" remains as meaningless as is "Education" as long as I am lacking any vision on what is [illegible] be learned and taught and in what spirit. The "M" accounts for the subject matter and the vision on M for the spirit. But M is only a paradigm and it is worth as much as it is as a paradigm. PME should prove its right to exist by being the best paradigm of psychology in education.

B. LEARNING THEORIES

1. *GENERALIZATION*
2. *PSYCHOLOGICAL THEORIES*
3. *NEUROPSYCHOLOGICAL THEORIES*

AGAINST GENERALISATION: MATHEMATICS, STUDENTS, AND ULTERIOR MOTIVES

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Faculty of Mathematics
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> "To generalise is to be an idiot. To particularise is the alone distinction of merit. General knowledges are those knowledges that idiots possess."
>
> William Blake (1757 - 1827)

The starting point from which I want to set out is a very common one in mathematics, namely that of a teacher giving an example. The question I wish to explore is what is the purpose of going through an example, although there are related questions of interest such as what is involved in recognising a counterexample. The proposals I shall make involve the balance between the general and the particular, but also how the persuit of the general can distort the particular.

I was struck by the term 'example', whose common structural useage requires it to be an example _of_ something. (To refer to a mathematical situation as an example or a counterexample is a claim for or an award of a particular status (see Lakatos, 1976 for techniques of resistance _against_ such claims).) We do examples to illustrate, embody or make concrete the general theory, thus indicating the subordination of the particular to the general. This practice also promulgates the view that mathematics is about general theories rather than particular problems, whence the former arose in the first place. We are therefore faced with the fact that we are doing the example because we are interested in addressing or approaching the thing of which it is an example, thereby framing figure and ground.

My concern is that students' perceptions of why we are doing examples may differ. Students have far less experience even with the particular situation under discussion (and they are often unaware there are others) which, as a consequence, absorbs all of their attention. The student sees only the particular (which is still quite general ie. not mastered).

For us, many examples are seen as equivalent, as generic instances of a theory or technique we may be attempting to illustrate. All the while we are seeing, andperhaps commenting on aloud, the general in the particular (to use John Mason's telling phrase) which our chosen example embodies for us. The ulterior motive of our only wishing to exemplify the general detracts from the individuality, the idiosyncrasy of any particular example. By always evoking, calling attention to the general, the enduring features, we miss out on the transitory, the temporary, the unique which may be of equal surprise and interest.

"There is one universal functioning without which nothing is noticed. This is the stressing and ignoring process. Without stressing and ignoring, we cannot see anything" (Gattegno, 1970). He goes on to add, "Nobody has ever been able to reach the concrete. The concrete is so 'abstract' that nobody can reach it. We can only function because of abstraction".

In the early seventeenth century many mathematicians worked on discovering properties of the cycloid (Roberval, Fermat, Galileo, Huyghens, Descartes, ...) eg. its arc length, the area under one arch, its volume of revolution etc. To us, this is but an example of their interest in questions of mensuration and the emergence during that period of general methods (eg. the method of tangents) for dealing with curves. However, it may be that the locus of interest then lay more with the particular curve in its own right, as a mathematical object worthy of study for its own sake, rather than as a particular example of a more general technique. Osserman (1981) replied to Dieudonné's Berkely claim of the inevitability of the prevailing direction of mathematics towards greater complexity and generality (contended with only by greater abstraction) by citing current examples of particular equations (eg. the Korteweg-de Vries equation), being studied deeply which can be seen as a modern-day counterpart of the cycloid.

There is also an aesthetic which prefers the tyranny of the general, perhaps as a reaction to the fear of special cases, to the idosyncrasy of the particular (eg. the initial unease at the 'sporadic' simple groups). The desire is for general methods and arguments whose power arises from the range of applicability, which is also therefore a testament to their indiscriminateness. I have seen an article using Sylow's Theorem to prove Wilson's theorem, which was rightly described as using a sledgehammer to crack a nut. There may be some mathematical interest that it can be done (yet another notch carved in the ever-widening belt of Sylow) - but again the particular is destroyed, those unique characteristics of the problem flattened. A question I will persue at PME is whether it is the steamrollering effect which arises precisely from the use of algebraic language as an expressor or conveyer of generality with the resultant loss of meaning.

The general can be seen as a mental template against which putative examples are found appropriate (?) and the more transparent and close the fit the more generic the instance. But also the more formless perhaps, devoid of individuality and interest, in a situation of rampant egalitariansim where every instance is as good as any other. The actor's face, able to stretch to take on expressions of different personnae, when relaxed in its natural state often seems expressionless - the price of its flexibility? The vacuity of set theory at all levels and, more recently Category Theory as the proposed agar for mathematical activity testifies to this.

It is possible to take an alphabetic analysis course where the elements are all called f and g, and a particular function never enters. Halmos has elsewhere told of the Ph.D. student oral who was unable to specialise to a single instance of the class of Banach Algebras on which he had expended a large amount of study. To specialse from the general requires a knowledge of the particular as well as the general. If a problem is worth studying, it is worth doing for its own sake. If text-book examples were justified on this basis a lot of tivial illustrations (a term we might usefully use to distinguish strong from weak, or generic from non-generic examples) would disappear.

The foregoing also raises questions about a particular type of example, classified by their function, namely counterexamples.

In a recent article, Desmond McHale (1980) recently drew attention to the paucity of actual counterexamples offered in mathematics courses, used to disprove converses of theorems or to show the necessity of certain conditions. The prime example McHale cites is $x \to |x|$ as being the only continuous, non-differentiable function. What may be happening is that the lecturer is perceiving the counterexample in a very different way from the student. To the presenter the counterexample is generic, that is it speaks of a whole class of functions, $x \to k|x + a| + C$ at the very least. For the students, however, it is a particular example. They see it not as a class, but as a single, specific function. (See Mason & Pimm, 1983 for more details.

Students may frequently be uncertain as to the role and nature of such a counterexample. Not only is it particular rather than general, but in the main they are not clear about the general statement it is defeating. Seeing one strange example often has little effect, and students here are in good company. It is often claimed that just one counterexample disproves a conjecture, yet the history of mathematics provides evidence that this is false. A single counterexample to a putative theorem is often incorporated into the statement of the theorem as the sole exception. Another approach is to alter the definition, or in other words to refuse to accord it the status of a counterexample. Thus the specific function $x \to |x|$ is excluded from the scope of the theorem (see Appendix 1) or students deny that it deserves to be called a function at all. Perhaps due to the air of artificiality which accompanies many specifically-contrived counterexamples, many students take this latter view.

There are a number of outstanding questions of note to be pursued:

Questions

> How can you discern the extent of the generality perceived by someone else when you are looking at a particular example together?
>
> How can you expose the genericity of an example to someone who sees only its specificity? Apart from stressing and ignoring, and repeating the general statement over and over, how can the necessary act of perception, that of seeing the general in the particular, be fostered?
>
> What is involved in recognising a counterexample?

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Appendix 1

A twelve-year old was interviewed working on the area and perimeter problem at last year's conference involving exploring the validity of presented generalisations such as:

> 'For any rectangle there is one with the same area and a larger perimeter'.

He was invited to come up with a similar generalisation. On the basis of three examples the result "the area is always less than the perimeter" was proposed. Another example was tried which didn't work, so an extra clause was tacked on, "when the area and perimeter are less than 20", to exclude this newcomer from consideration.

Note: I am grateful to John Mason for some of the ideas discussed in this paper.

MODELS OF THE PROCESS OF MATHEMATICAL GENERALIZATION

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The guiding question

This paper is based on work carried out by a project team at the University of Klagenfurt. The central topic of this project is mathematical generalization and the cognitive processes producing generalized concepts and methods. Formulated as a problem: Which are the cognitive activities and their means and instruments which result in the collective and/or individual construction of generalizations? Related questions are those concerning transferability and flexibility of knowledge: How do we learn from our activities? How do we as thinking subjects recognize that a given task has the same type or structure as one which we have solved already before that?

A process model of mathematical generalization

The model presented here is based primarily on an analysis of existing research findings of theoretical and empirical nature (Rubinstein, Dawydow, Lompscher; Piaget, Aebli; Krutetskii). Very helpful were also thorough studies of epistemological properties of mathematical knowledge. The model will be described by the use of five states or stages. These not necessarily are the phases of the process in a chronological order. The process is rather a complex network of these states. The cognitive activities corresponding to the states should be imagined not only sequential but also parallel whereby in the course of time in certain phases different activities are more dominant than others. So there is a dialectic interdependence between the stages and only for the sake of clear description and subsequent empirical investigation each appears here in an isolated form. We hold the opinion that in the individual some stages (probably those with higher numbers) might be undeveloped, unconscious and therefore not resulting in directly observable behavior. This depends to a great extent on the cognitive means which the individual disposes (linguistic tools for expressing arbitrariness, or strategies of thinking). Finally the model does not assert to describe any kind of innate abilities or a factual, empirically testable process. Rather it is meant as a general intellectual method for obtaining generalizations which can be and has to be acquired and developed like other mental or manipulative techniques. The five states are the following.

State 1: Recognition and estallishment of relationships

When confronted with the description of a situation which explicitly or implicitly contains a problem the first state is analysis of the situation. The goal thereby is the detection or construction of relationships between numerical (or geometric) features and magnitudes of the objects given by the description. These might be equations, inequalities, relations between sets, functional relations or limit relations for instance. These relationships usually will contain also unknown values of the magnitudes. We have so far only considered numerical magnitudes. This state results in a concrete model (without using variables) of certain mathematical aspects of the given situation (which in itself already might be a mathematical one).

State 2: Finding situations with the same relationships

The second state in a sense tests the importance and value of the relationships detected in the first state by looking for other situations which give rise to the same or very similar relationships. This state not yet does make use of the general structure of the relationships but rather of the character of the magnitudes connected by the relationships (for instance additivity of a magnitude like length, weight or monetary value) The objects which carry the magnitudes as their features will vary depending on how flexible the knowledge about the magnitudes is. Here activities of state 5 type are relevant but with respect to other relationships or concepts than those which are the objects of the actual process.

State 3: General representation of the relationships

The relationships (so far between numbers or values of magnitudes) are now represented by the use of different kinds of variables (letters, geometric variables, verbal expressions, drawings). These variables are of a referential, descriptive and semantic character in that they denote and stand for the values of the magnitudes of the former states. Nevertheless this is the first step of a process which will transform the found relationships from the description of aspects of a situation to objects of thinking. The emphasis is shifted from the analytic character of states 1 and 2 to synthetic activities.

State 4: Formal relationships

This is the transition from content to form, from semantic to syntactic variables. The relationships and their representation remain essentially unchanged but their meaning is deeply transformed. The representation no longer denotes certain relationships but is considered as expression of the general form and the structure of them. So it is a shift of point of view. It is possible that this transition applies only to some of the occurring variables and that others remain semantic ones or even continue to denote fixed values of magnitudes.

State 5: Operations with the formal structure

In this state manipulations with the formal expressions of state 4 are carried out (e.g. transformations), formal consequences are drawn and the formal structure is applied either to solve new problems of the same formal type or even to construct problem situations of this type. Especially this is therefore a "recurrent" state.

Questions posed by the model

The states of the model are characterized by the quality of the products of the intellectual activities dominating in the respective stage. This now poses questions the answer to which will make the model an operative one:

- Which are the intellectual/cognitive techniques and strategies which produce these products? More concrete: Which mental activities are carried out by the individual in a certain state?
- Which are the tools of the cognitive activities in each state? These can be for instance: letters as variables, drawings, imagined manipulations, earlier learned mathematical concepts.
- Which are the means (like verbal, iconic, symbolic) appropriate for expressing and communicating the results of a state?

Since our work is mainly motivated by didactical interests further questions to the model are:
- Can the cognitive means and tools necessary for such a process be taught successfully, and by which methods?
- Which states are exhibited by students educated by existing teaching methods?
- Does the usual system of school mathematics possibly exert a detrimental influence on student performance in one or the other state?

The topic of the interviews

To find cues for answering these questions interviews were made with persons of different age but mainly with students in grades 7 to 9. The content and form of the interviews was changed and refined stepwise according to obtained results and to new theoretical hypothesis. The essential tool in all interviews was a word problem (sometimes not formulated as a problem but in the form of a story with complete data). The general type of it is represented for instance by the following (it was motivated by interviews documentated by Krutetskii): In a tennis club there are k courts and at a given time p persons want to play. How many singles and doubles, resp., will there be?

Typical questions in the interviews are: Can you find relationships between the numbers given in the text? Can you write them down? Can you describe them verbally? Can you find other numbers which make the story meaningful? Which numbers can be calculated from others and in which way? What is the general pattern? Can you devise other stories with the same pattern? How would you tell somebody else about this general pattern? Different texts of the same formal structure were presented and the subjects were asked to analyse them in the same way as the first one; comparisons were urged. In the last version the most effective question was: Imagine you are taking care of the tennis courts, how would you distribute the players (numerically)?

Empirical findings related to the model

First, it should be remarked that no general or statistical validity is asserted. It is only reported which phenomena have been observed in our interviews.

- There are two kinds of relationships between the (explicit and implicit) numbers in the text: Static ones (like $d+s=k$, $4d+2s=p$, d-number of doubles, s-number of singles) which can be read off the text by using just the meaning of arithmetic operations; dynamic ones which usually are detected only by the use of some manipulation (actually distributing the players in the most natural way gives $p-2k=2d$). The static ones numerically describe the situation, the dynamic ones solve a possible problem related to it.
- Most of the interviewed subjects by themselves did not devise or imagine actions possible in the described situation to detect relationships so that only the static relationships were found (but these were established in all cases and even with some degree of generality and recognition of their formal structure).
- The above cited question (putting the subject into the context of the situation!) prompted with great success at the subjects imagination of actions which usually lead to the dynamic relationships. So it appears that for the subjects a mathematical problem by itself does not relate to any actions to be carried out by the individual himself. But on the other side even those with poor performance at school were able to act and to solve thereby the problem. It was also observed that knowledge of standard mathematical methods (linear equations) even blocks devising other ways of solving a problem (like imagined actions).
- The interviews were made without any prior instruction so that the subjects had only training in usual school mathematics. Under

these conditions they showed successful behavior corresponding to state 1 (see above), state 2 and partly state 3; state 4 was missing (and consequently state 5). We had not available any means to prompt this state (this really is an open problem).

- Having only the static relationships at one's disposal (in the form of linear equations) one rather rarely establishes the structural equality with contextually different situations before having derived the equations in the cases to be compared. After having devised appropriate actions (like distributing) structural equality is recognized before solving the problems as long as the form of the action is meaningful or applicable (at least after some slight modifications). This recognition even supports the solution of the new problem. One restriction must be made: the actions sometimes were not recognized consciously in all their constitutive parts and the essential general form of the action (or the relationships created by it) was not established. This led to failure or mistakes in situations with different numerical (and textual) data.

A hypothesis

But summing up, a hypothesis could be: relationships (like formulae, equations) have a higher degree of transferability if they are derived from actions by the individual. Not the generality or formality of the representation of relationships in itself is important for flexible transfer but the way in which the student obtains the representation (linear equations usually are offered by the teacher and only memorized by rote learning). Also the chance to detect more implicit relationships is higher when carrying out appropriate actions.

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About the mental action leading from the special case to "The General case" (Generalisation: When, Why and How?)

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Generalisation is passing from the consideration of a given set of objects to that of a larger set, containing the given one.

In this note generalisation does not stand for the technique (e.g. removing a restriction, or replacing a constant by a parameter) but for the mental action behind it. We would like to understand "When, Why and How"a profesional mathematician generalises the concepts subject to his research.

Mathematicians need not answer these questions. For them a generalisationis the consequence of a deeper understanding of their subject. The mathematician gets the insight that the problems he is considering are becoming clear in a more general context (sometimes generalisation leads to the solution of these problems).

The difficulty in describing this mental action, lies in the fact that mathematical theories which generalise known concepts are always presented in a final form. A large number of actions have taken place before this form is obtained.

Therefore we worked out formal principles to distillate from mathematical theories those concepts and properties which are the result of "generalisation".

The program of secundary school mathematics is mostly delivered in a final form. There are no or very little possibilities to obtain general concepts in the natural way they are found (at best the teacher indicates this natural process). It is our aim to use the above mentioned principles to point out when, why and how "generalisations" should be made by the students.

1. Mathematical mental actions.

The study of "generalisation in mathematics" is only one part of a large research program on mathematical actions, aiming to make up an inventory of those mental actions which are both fundamental and specific to the creative work of the mathematician.

cfr. (2,3.4).
In order to obtain a unified approach to study the different actions occuring in the work of mathematicians, the following definitions of mathematical action is used:
The subject of activity is situated in a so called "extrapolating sheme". The original objects and problems are ordered with respect to their properties in such a way that using only logical rules some of the problems are solved (i.e. Theorems are proved.). During this proces the properties become more or less independend of the original objects, of their original meaning. So we have shematised the subject. However we also demand that the resulting sheme allows intrinsic reasoning. Such a sheme is called an extrapolating sheme. At this point we like to draw attention to an important fact, although, as we saw, the original meaning of abjects is lost in the extrapolating sheme, there is some feedback, the results obtained by the mathematician (read mathematical action) do give more insight in the problems he started sith.
We are now defining mahtematical action as follows:
A mathematical mental action is a mental action on an extrapolating sheme or between extrapolating shemes. (cfr.(4))

2. Generalisation. An example.

We give a rough description of an example which turned out to be very important in the history of mathematics.
In órder to obtain a solution for Fermat's last theorem , mathematicians where lead to consider the unique factorisation property (UFP) of integers for more general numbers as well. Namely for numbers which are combinations of integers and roots of unity. It turned out that the UFP did not hold in this general case. However the work didn't stop at this point. In two different ways Kronecker and Dedekind generalised the concept of UF itself.
This was the starting point for a whole new theory. Although the original problem was not solved completly Kronecker did obtain a lot of partial results which proves that there is more insight in the original question. (cfr.(1))

3. Generalisation.

We now try to give a description of what we call generalisation. We will indicate the corresponding properties in the above mentioned example.

In order to say that a mathematical theory A consisting of objects, concepts, theorems is generalised to a theory B we demand:

I. That the different parts of theory A have a meaning in theory B. Objects and concepts from theory A occure also in B, at least in an analoguous way.

e.g. Integers, prime numbers become integral complex numbers (combinations of integers and roots of unity), irreducible numbers, prime irreducible numbers in Kronecker's generalisation. In the theory of Dedekind tey become ideals and prime ideals. The theorem of unique factorisation of integers is replaced by the theorem on unique factorisation of ideals in the Dedekind approach.

II. The original theory A changes after the generalisation (the feedback). One gets more insight or one solves some problems posed in theory A. The classification of the objects in A is easier the theorems in A are obtained as corollaries of theorems in B.

e.g. It is possible to solve some cases of Fermats last theorem using the general theory, although this theorem is stated completly in theory A.

The difference between prime nimbers and irreducible numbers gives more insight in the concept of prime numbers.

III. Theory B has more ramifications compared with theory A.

A lot of new problems can be considered.

Theory B can be applied to more subjects.

It can be extended in different ways.

e.g. Dedekind's theory was the beginning of what is now called commutative algebra. This has applications in a lot of areas also outside number theory, the main example is of course algebraic geometry.

Number theory itself is changed to. The main problems of classfield theory can only be staded in the general context.

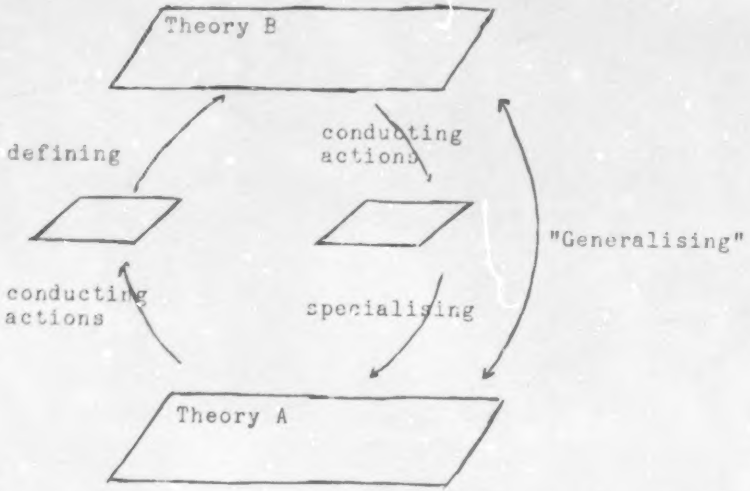
The first criterium (I) is close to the technical meaning of generalisation. It is well known.

The II an the III-th are more important in our view. They are related to why and how one generalises a theory.

Demanding them is avoiding wild generalisations which do not add anything essential to mathematics. This leads to theories without depth.

They also show that in order to descripe a generalisation proparly, one has to consider what we call " conducting actions ". Generalisation is not possible without some other mathematical mental actions as well.

In terms of the extrapolating shemes we become:



A more deeper study of these is our main subject of research at the moment. We hope to report on it at some other occasion.

4. Generalisation in the classroom.

The above consideration mainly apply to mathematical research. We can draw some conclusions to what should be done in a classroom. Although this has to be worked out more carefully we mention some points already.

" Why " generalising in mathematics. This question is certainly put forward by the students. The teacher has (at least) to point out that the problems posed in theory A, but not solved, are the starting point of a generalisation. Students should be made clear that getting more insight in what one knows already, is what you try to do when you make a new theory.

In the beginning theory A is the most important one, theory B is only a help.

In the classroom it might be usefull to consider different possible generalisations. The criteria especially II and III, can be used to chose between the alternatives. The better they are forfilled the more usefull the new theory will be.

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CYCLES OF LEARNING AND THE SCHOOL MATHEMATICS CURRICULUM

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Several different groups of researchers (Biggs and Collis, 1982; Marton, 1981; Case, 1980; Fischer, 1980) have put forward similar models for the development of intellectual functioning in children and young adults. Basically the models incorporate major stages which are largely determined by the kind of elements that the individual can use and by the nature and complexity of the operations which he can apply to those elements. Examples of the nature of the elements and the kind of manipulations available at each of the stages are as follows:

Sensori-motor: the elements are the objects in the immediate physical environment, and the operations involve the management and co-ordination of motor responses in respect of these objects.

Intuitive/Pre-operational: the elements become signifiers (word, images, etc.) which stand for objects and events, and the operational side involves the manipulation of these in oral communication.

Concrete Operational: the elements develop from mere signifiers to concepts and operations which are manipulated using a logic of classes and equivalences, both elements and manipulations being directly related to the real world.

Formal operational: the elements are abstract concepts and propositions, and the operational aspect is concerned with determining actual and deduced relationships between them; neither the elements nor the operations need a real world referent.

The structure of the learned responses which occurs within each stage becomes increasingly complex as the cycle develops. Uni-structural responses represent the use of only one relevant aspect of the mode; multi-structural, several disjoint aspects, usually in a sequence; relational, several aspects related into an integrated whole; extended abstract takes the whole process into a new mode of functioning.

These notions are well summarized in Figure 1:

Figure 1

RESPONSE MODEL OF INTELLECTUAL FUNCTIONING

Mode (Developmental Stage)	Response Structure (Learning Cycle)		Use of Language
Sensori-motor (infancy)	Unistructural Multistructural Relational Extended Abstract		
Intuitive/Pre-operational (early childhood - pre-school)		Unistructural Multistructural Relational Extended Abstract	Words
Concrete Operational (childhood to adolescence)	Unistructural Multistructural Relational Extended Abstract		Sentences
Formal - 1st order early adult		Unistructural Multistructural Relational Extended Abstract	Propositions
Formal - 2nd order and higher order adult	Unistructural Multistructural etc.		Propositions of increasingly higher order of abstraction

Three crucial aspects of this model are, (1) within each stage the levels of functioning develop in an hierarchical fashion with the higher levels subsuming the lower levels, (2) the highest level of one stage (or mode of functioning) becomes the lowest level of the cycle for the next stage and (3) the movement from one stage to the next represents a much more significant and difficult development than does movement from one level of skill to another within a given stage.

Schools mathematics is concerned with the cycle of learning in the concrete operational mode and thus we expect that both the elements and the operations involved in reasoning will be directly related to the empirical world. This will be reflected in the need the child has to "close" arithmetical operations in order to make sure that a unique empirically verifiable result is available as a result of his reasoning. Elsewhere (Collis, 1975) the writer has described development through this cycle as a growing ability to hold off "closing" for longer and longer periods. The sequence may be illustrated by responses to the following item.

Find the value of Δ in the statement:
$(72 \div 36) \times 9 = (72 \times 9) \div (\Delta \times 9)$

Prestructural responses
"Have not done ones like that before, so I can't do it."
"Don't want to do it."
Both respondents indicate that they are unwilling to engage in the task.

Unistructural responses
"36- because there is no 36 on the other side."
"2- because $72 \div 36 = 2$."
Both responses take only one part of the data into account. The first response shows a low level "pattern completion" strategy; the second response shows one closure and then an ignoring of the remainder of the item.

Multistructural response

2×9	$648 \div (\Delta \times 9)$
$= 18$	$648 \div ? = 2$ that is, 324
	looking for 18 (2×9)
Hence 324	

This response incorporates a series of arithmetical closures to reduce the complexity and to focus on "Δ". However, the student appears unable to keep the overall relationship in mind throughout the closure sequences and ends up getting lost in a "maze" of his own creation.

Relational response

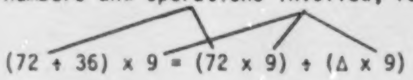
2×9	$648 \div (\Delta \times 9)$
$= 18$	$648 \div 9 = 72$
	then $72 \div 4 = 18$

Hence 4

This response also involves a sequence of arithmetical closures but the student is able to keep the relationships within the statement in mind and thus successfully solve the problem.

Extended abstract response

First step involves obtaining an overview of the relationships between the numbers and operations involved, for example:



$$(72 \div 36) \times 9 = (72 \times 9) \div (\Delta \times 9)$$

The pattern suggests something akin to the "distributive" property - this hypothesis is tested out thus:

$$\frac{a}{b} \times y = \frac{a \times y}{b}$$

This immediately solves the problem (without necessity for closure) as follows:

$$\begin{aligned} & (72 \div 36) \times 9 \\ = \quad & (72 \times 9) \div 36 \\ = \quad & (72 \times 9) \div (4 \times 9) \quad \text{Hence } 4 \end{aligned}$$

This response shows the following characteristics:

1. Focusing on the relationships between the operations and the numbers rather than regarding the operations as instructions to close;

2. an hypothesis suggested by the data is set up;

3. avoiding closures wherever possible as these change the form of the statement and "hide" the original relationship.

From this example it can be seen that successful achievement at the relational level is not a low level achievement. Although it involves simply closure techniques and, on occasions, the ability to use generalized numbers it does require the student to maintain control of the inter-relationships within the statement(s) given. In fact achievement of this level is sufficient for most students going on to tertiary level courses (such as typical Arts subjects, psychology, medicine, law, sociology etc.) where high level ability to manipulate mathematical symbols per se is not normally required. It should be noted also that achievement at the relational concrete level is a pre-requisite for moving to the formal stage; this is a necessary but not sufficient condition however. Movement to this new stage is a vastly more difficult step than the earlier movements through the levels of the concrete stage because it involves the ability to overview a total abstract system including its internal relationships and its relationship to other systems.

Let us now turn briefly to the reasons for teaching mathematics to virtually all children in school and then examine them in the light of the background set out above.

There appear to be three main reasons why an academic subject such as mathematics is included in school programmes: to socialise students; to develop logical functioning and to train specialists.

To socialize students. All students, regardless of specific interest in mathematics, need to become acquainted with certain basic mathematical content in order to function effectively within a society that rests so much of its decision making on mathematical models and calculations. Citizens in such a society need mathematics to be able to manage their own affairs and to be eligible for even semiskilled occupations. At a higher level it is desirable that they also understand the kind of mathematical modeling and calculating that lies behind government, business, and trade decisions.

To develop logical functioning. This might be seen as the prime developmental task of the school-aged child and one which the school is uniquely organized to foster. Donaldson (1978) sums up an enormous amount of evidence to make two points. First, "disembedded" thought, the ability to solve problems involving propositions and structures without dependence upon a concrete reality, is perhaps the most highly prized skill in society. Second, the aim of teaching certain fundamental skills such as those involved in elementary mathematics in schools must be seen as fostering the development of the child's power of logical functioning not, as was supposed in the past, the acquisition as soon as possible of adult-level skills. Mathematics is uniquely placed to foster the growth and development of logical functioning because of its very nature. It encourages and practices the use of "disembedded" thought right from the earliest days in elementary school through to the end of schooling: the use of numerals to represent numbers of elements in a set is an example of this idea at the lower elementary school level. The beauty of mathematics in the context is, of course, that one can see an hierarchy of abstraction extending from the use of small numbers and concrete operations at the lowest level to the use of variables and defined abstract operations at the higher levels. In addition mathematics has a logical structure that can be discerned at a very elementary level, (e.g., the relationship between the addition and multiplication operations) or at a highly sophisticated level (e.g., the application of the field axioms in solving equations).

It goes without saying that, to have an effect on the child's cognitive development, we must be able to give the child practice with a variety of content that he can understand and manipulate at his current level of functioning. Here again mathematics is uniquely placed. Peel (1967) has classified the cognitive functioning of the concrete operational child as the logic of classes and differences; equivalence and substitution. This means that these children are able, within the constraints of their empirical reality, to classify and reclassify, to rank order, to see equivalences, and to make appropriate substitutions. These skills are exactly the ones required to handle the mathematical ideas associated with the notion of sets and operations on sets. The author (Collis, 1969, 1975) has shown how virtually the whole of the elementary-school mathematics course can be based on this one mathematical notion that is congruent with the child's logical development.

In short, if fostering the development of logical thinking is one of the main aims of education, mathematics is a study fundamental to achieving that end.

To prepare mathematics specialist. This intention is meant to include both those few students who have the ability and interest to become professional mathematicians and that larger number of students whose future careers are going to be in fields where manipulating mathematical type relationships to obtain new insights is more and more forming the basis for decision making. These students will need to immerse themselves in the content-process aspects of mathematics to a much greater depth than those who merely wish to satisfy the social and intellectual skills aims previously outlined.

If we examine the reasons for teaching mathematics in relation to the model of intellectual functioning put forward earlier in this paper it would seem reasonable to deduce that many school systems have been totally unrealistic in setting leaving expectations in mathematics for the majority of students. The expectation that all students, by the end of junior high school, would be able to handle variables and manipulate abstract systems is a typical example. Two points that are clear from analyses of mathematical material and achievement in this context. First, the difference between the relational level of response and the extended abstract response in a mathematical topic is enormous and it requires a prodigious effort on the part of the individual to achieve the higher level. This implies that the individual is highly motivated to work with the kind of material, structures, and logic peculiar to mathematics. Even if we leave aside persons with a low level of general intellectual development and include only those who are well advanced in this area of development there is no prima facie reason why they should be so motivated. The second point is that the general population does not require an ability to respond in mathematics beyond the relational level in order to be able to manage quite successfully in their adult lives even if this includes achieving certain tertiary professional qualifications.

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HOW TO PROVE RELATIONAL UNDERSTANDING

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Abstract. *We distinguish two modes of mathematical activities (syntactical versus semantical), two levels of skills (instrumental versus conceptual), and three levels of understanding (instrumental, relational, communicable). The main issue will concentrate on the development of relational understanding and on possibilities to prove relational understanding without stressing "formal" aspects of communication.*

1. Syntactical and Semantical Activities

There are at least two possibilities to solve the problem "$\frac{1}{2} + \frac{1}{3} =$ ". Student A remembers "rules" and answers "$\frac{2}{5}$", because he took the wrong rules. Student B remembers pan cakes and answers "$\frac{7}{8}$", because he had a wrong "imagination" of the total "one half plus one third". (Of course we also have students who answer correctly. But incorrect answers give more insight upon the process of thinking.)

Student A worked syntactically. He manipulated symbols according to rules or algorithms. There are many examples for syntactical activities in mathematics education besides computation of fractions: Working in different place value systems, paper-and-pencil algorithms for the four basic operations, pressing sequences of calculator buttons, replacing variables by numbers in a formula, transforming formulae, differentiating, computing integrals, ... "Reproducing knowledge" also can be done syntactically: Repeating basic facts of the addition or the multiplication table, "citing" definitions or theorems or proofs, "selecting appropriate" formulae or algorithms or procedures, ... Working syntactically often is the result of a stimulus-response-learning.

Working semantically however needs the knowledge of mathematical relationships. Semantical activities are arguing, deducing theorems, developping proofs, inventing or deducing rules or procedures, finding unusual but correct solutions of a problem, identifying variables or algorithms, ... Working semantically only is possible if the student "understands" the problem. But!

2. What Does "Understanding" Mean?

According to Skemp [7] we first distinguish instrumental and relational understanding. A student has "instrumental understanding" of the problem if he can select and apply appropriate rules to solve the problem, but he does not know why (Skemp: "rules without reasons"). Activities which depend on instrumental understanding we call "instrumental skills". It is obvious that many of the syntactical activities mentioned above often are instrumental skills.

Relational understanding is more: "knowing both what to do and why" (Skemp). It is "the ability to deduce specific rules or procedures from more general mathematical relationships". Relational understanding is sufficient for being convinced oneself, it is less than being able to convince other people. Relational understanding does not include aspects of why or how the process of deducing was started or what modes of presentation or communication are used. Working semantically needs relational understanding.

It would be too simple identifying instrumental understanding with syntactical activities and relational understanding with semantical activities. A student for example, who solves the multiplication problem 325x417 in the following way without mistakes and without hesitating or breaks for thinking

	325x417
(a)	1300
(b)	325
(c)	2275
	......

and who is able to do that with similar problems in the same way, such a student works syntactically. He has the "skill" to multiply.

Analysing this skill we have to distinguish two possibilities.

It may be that the skill has been mechanical all the time. The student can not explain why he worked in that way. He does not know if the adding of zeros in the lines (a) or (b) or if permutations of the lines (a)-(c) are allowed. He just has an instrumental understanding: his skill is instrumental.

Otherwise that skill also might be the result of a mechanized ability. The student can - anytime when wanted - explain his doing and discuss correct or incorrect writings. He understands the concept of that algorithm, he has the relational knowledge for his skill. Therefore we will call it "conceptual skill" (higher skill in [5]).

Summarizing we can state, that syntactical activities are possible as instrumental skills (without relational understanding) or as conceptual skills (based on mechanized relational understanding). Therefore tests with emphasis on syntactical activities are not very helpful to prove if the student "understands" the problem.

3. How to Prove "Understanding"?

We defined relational understanding as the ability to deduce specific rules or procedures from more general mathematical relationships, excluding aspects of why or how the process of deducing was started or what modes of presentation or communication are used. With the words from Greeno [2] relational understanding is the well "connected" and "coherent" internal representation which "corresponds" to the mathematical problem, theorem, concept, ... Relational understanding in our meaning does not depend on special performances, on demanded modes of action, on the usage of general agreed phrases, procedures, algorithms, or symbolisms.

When we ask explicitly also for certain well defined procedures, abilities or skills we ask for more than relational understanding. We then emphasize also the process of communication. Mistakes or wrong answers may originate in this case also from "misunderstandings": the student did not understand the question or the student gave an answer which was not "understood" by the teacher because it was too individual (far away from mathematical conventions or agreements or not using the expected mathematical language at all). We here discuss a higher level of understanding which we called communicable understanding (see [5]). Formal, logical, or symbolic understanding (Skemp [8]) are special aspects of communicable understanding.

We observe the distinction of relational understanding versus communicable understanding in "the concept of premathematics" (Semadeni [6]), in "proving pre-mathematically" (Kirsch [3]), in working with manipulatives, or in a non-verbal introduction of mathematical ideas. In all these teaching modes the concept formation is separated from the formal aspects of a mathematical communication.

The distinction is also necessary for diagnostic purposes. Almost every teacher remembers situations, where he got "meaningless" answers though the student "obviously" has had the appropriate understanding.

Mathematics instruction in our country strongly concentrates on communicable understanding. But many test results show that we often produce only instrumental understanding. Again, tests with emphasis on syntactical activities are not very helpful. To prove "understanding" we need tests on semantical activities. If the student succeeds he has communicable understanding. If he fails we may interpret his answers. Often they will tell us if he at least has a relational understanding. Additional interviews may help to clarify.

4. How to Develop "Relational Understanding"?

Relational understanding is the entrance key for being successful in the mathematics classroom. Relational understanding is an inevitable condition for the development of conceptual skills, for working semantically and for the development of communicable understanding. But relational understanding mostly is intuitive and unconscious. There is no easy reflective way to teach it.

Each student must "construct" internally his own relational understanding. He has to re-invent ideas and relationships. He needs a strong correspondence between external mathematics (examples, tasks, problems, concepts, procedures, ...) and his internal representation. The more feed back he has and the more frequent the interaction is, the better is the "correspondence" (Greeno [2]). We see the following possibilities to further more relational understanding.

- Give many examples and counter examples before any systematization. Help students to generalize more intuitively.
- Allow trials as a well accepted mathematical method for working successfully.
- Let the students "construct" mathematical relationships systematically by guess-and-test procedures (one-way-principle, see [4]).
- Stress estimations.
- Give more freedom in using words and notations. Let the students invent own formulae, names and notations and let them discuss which one could be the best. Only lateron inform them how the mathematicians decided.
- Reduce syntactical working.

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FORMS OF KNOWLEDGE IN THE CHILD'S LEARNING OF MATHEMATICS

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INTRODUCTION

In considering what the pupil has to know to be successful, an account of schooling expressed solely in terms of the formal units of the curriculum is conceptually inadequate. An alternative approach to the exploration of school success, or failure, is to attempt to characterize the critical competencies that the pupil has to develop and use in order to cope with the demands of school. For example, Mehan (1980) has proposed that in order to participate effectively in the classroom pupils need to syncronize two forms of knowledge, one covering academic content, the other interactional form. He argues that it is not merely enough to have mastered some parts of a subject-matter, the pupil must also, at least, appear to conform to the rules that govern the presentation of such knowledge in the classroom.

The following is an account of a part of the information derived from a study of a Teacher and her pupils in a special class for children with learning problems. One goal of the study was to analyze the role of different types of knowledge in children's mathematical performances. This report will focus principally on the special role of the child's metacognitive knowledge. The forms that such knowledge take in mathematical problem-solving include checking, predicting, and estimating. They refer to the basic characteristics of thinking efficiently in learning situations (Flavell, 1979 and Brown, 1980). Another goal was to provide some information on the extent of the teacher's influence on pupils' performances. For example, the pupil might on some occasions operate with a "formulaic" strategy in school learning where much of the material is memorized with minimal understanding. By contrast, the pupil might on other occasions adopt a "generative" strategy in an attempt to comprehend the relations between the elements of a classroom task. The pupil using a formulaic strategy may be taking up (or have had thrust upon him or her) a dependent learning role, so that the pupil simply acts as a reproducer of the teacher's ideas, values, and problem-solving strategies; whereas the pupil using the generative strategy is likely to have an independent learning role, and to be an autonomous producer or, with the teacher, a co-producer of knowledge.

A STUDY OF METACOGNITIVE KNOWLEDGE

Information on the Teacher and children was obtained by observing in the classroom and through interviews. In the following segment of an interview with the Teacher, the topic of "checking" as an instance of the child's monitoring of work was pursued. She describes how she advises pupils to check work, and judges their fidelity in following her recommendations.

I: Do they check their work? Are they supposed to check their work? T: I have taught most of them how to check their own subtraction questions by adding the bottom number with the answer to get the top one. Basically it's just a visual check. "Six divided by two. I have six put into groups of twos. I'm not going to get three, er twenty eight groups. I'm not going to get eighteen and I'm not going to get twelve. It doesn't make sense. I have to get a smaller number." Just usually checking to make sure if it makes sense or not. "If mother had five cookies and she gave four away, how many is she left? She can't possibly have nine."

I: You were saying that they do this or they don't do this?

T: They're encouraged to do it.

I: And to what extent do they try to do that? T: I don't think that many of them do it. They simply get their work done and hand it in and so on to something else. Get some free time or whatever have you. It's simply "let's get it done." There are some that try to get their work right but they won't check to see if they have them right, or have answered all the questions. This happens quite often. I'll call them back and say "You didn't answer this question. You left that one out, you didn't do this one, what was the lesson?" "Oh, I didn't see those." So they didn't go back to see if they had everything down. "Did I have seven questions to copy off the board? Did

I copy seven down? Did I copy five down?"

I: So there's two kinds of checking that they are not always doing. One is just to see that you've completed the actual work, that you've put answers down or copied all the problems down that you were given, and the other is to see whether you've done the work properly, when you actually did it?

T: Right. If you allow them they just don't do it. They think basically "I've got to get my work done and handed in, let the teacher check it over. Then if I get the work corrected, all right!"

The Teacher's assumptions seem to be that pupils typically:

(1) Do know appropriate checking procedures to allow them to arrive at an estimate of the right answer, and thus can make progress toward the right answer.

(2) Do not check to see whether they have copied all of the work from the black board, or whether they have answered every question, but instead, leave it to the Teacher to check their work.

The following are samples from the interviews with two female pupils; Terry, 8 years, and Carol, 11 years:

I: Do you ever check your work? T: No. I try to, but I just make the same math questions more worser, so I just leave them alone. I: Oh, what do you mean that you make them worse? T: Like if I hand 'em in and I corrected them, right? Then she says "That one certainly isn't that." Say I handed that in and it was ten hundred, and she says "That wasn't it." I was wrong. I: Do you think sometimes you have them right and then you change them when you check? T: Well, I never check no more because I used to get them wrong. Now I sort of get them wrong the same way.

I: Do you ever check your work? C: No. I just look over it. Well when you're supposed to, like, um when Mrs. W. was here we, um did this kind of thing and she would put the answers up on the board and we would check them by ourselves and that's fun 'cause I like doing that. I: Oh I see, you mean you would check to see if your answer was the same as hers?

C: They would show the real answer, the right answer on the board. If you check over somebody else's work and you'd give, we did that last year and we passed over some. You take one person's work and you check over it to see if you got all your work right. You would take your own paper and check over.

Terry does not appear to share the values that the teacher attaches to checking, nor does she understand the basic procedures for checking. She has apparently tried repeatedly to arrive at the "right" answer in the past without success, consequently she has given up her mode of checking. Carol on the other hand, agrees that there is value in checking, but her definition is very different from the Teacher's. "Checking" to Carol simply means to compare her answers to the teacher's "right" answer, or to other pupils' answers.

The Teacher's view that the pupils have been taught and therefore know how to check their math work is not supported by the evidence; however, the Teacher's suggestion that her pupils leave her to check their work does receive some support in these two cases. She appears to underestimate the difficulties involved for pupils in her class when they are asked to check their math work. Successful checking (or monitoring) presupposes that the pupil is competent in basic mathematical operations such that errors can be detected, and that knowledge of correction procedures, including estimation and prediction, are available. These competencies even when developed by "learning disabled" children are often inaccessible to them, as their diminished confidence results in the adoption of the strategy of "playing it safe", exhibited in Terry's abandonment of her checking procedures and Carol's delight in using others' presumably "right" answers. The two pupils seem to be operating, at least in the math class, with a general formulaic strategy in learning.

THE CONSTITUENTS OF SCHOOL COMPETENCE

In developing a conceptual framework to make sense out of school experience, pupils operate simultaneously with three kinds of knowledge. The first type is academic knowledge, or grasp of the content of subject matter. The second is social-cognitive knowledge, which refers to the pupil's ability to make both meanings and intentions clear to others as well as to understand their meanings and intentions. As Erickson and Shultz (1981) suggest: "The production of appropriate social behavior from moment to moment requires knowing what context one is in, and when contexts change, as well as knowing what behavior is considered appropriate in those contexts" (p. 147).

The third kind, metacognitive knowledge, relates to the pupil's skill in monitoring his or her own performance by using feedback and corrective procedures. One example of self-monitoring is the phenomenon of the "retraced false start", where the speaker detects an "error" in his or her own speech and corrects it by substitution, deletion, or addition of new material (MacWhinney and Osser, 1977). Other equally common examples are where the pupil might monitor comprehension of a task by asking such questions as "What is this all about?" "What is the next step?" and "Did I forget anything?"

INDIVIDUAL DIFFERENCES AND SCHOOL COMPETENCE

Differences among pupils in academic performance can be better understood by referring not only to variations in academic and social-cognitive knowledge, as Mehan (1980) appears to suggest, but by considering also the likelihood of significant effects of differences in metacognitive knowledge. To return to the topic of the monitoring of comprehension, it is conceivable that a pupil who closely monitors his or her understanding of a mathematical task may select a more appropriate problem-solving strategy if it appears that progress is not being made. On the other hand, a pupil who monitors his or her performance in a casual manner may often miss the clues that indicate he or she is not on the right track, so that faced with undefined difficulties the pupil might be inclined to give up, as Terry did, rather than to persist with the work.

A possible hypothesis is that a pupil who typically employs the "close monitoring" strategy for a given set of school mathematical problems will be more likely to adopt the generative mode of learning and its associated, self-defined role of producer or co-producer of knowledge. The corresponding hypothesis is that a pupil who typically adheres to the "casual monitoring" strategy for a given set of school mathematical problems will be directed into the formulaic mode of learning and the role of reproducer of knowledge. The performances of the two children discussed earlier seem to fit this latter characterization. However, it is likely that every pupil will follow a "close monitoring" strategy for some sub-sets of problems and a "casual monitoring" strategy for others. If this is the case the pupil would likely vacillate from independence of the teacher to dependence on her as a function of the particular subject matter being taught and the specific context of learning.

To end this report, it seems likely that the examination of pupils' school performances in terms of the types of knowledge necessary for coping with the demands of school, and also the different conceptions of such demands by Teacher and pupil, will likely result in increased understanding of the pupil's school success or failure.

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REPRESENTATIONS IN MATHEMATICS.

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The first mathematical representations that everyone encounters are for objects to count:such as ancient people who enumerated sheeps by way of little stones,we learn in the first schoolyear how to use sticks,chips,marbles or even abacus'balls,each of them being substituted to one of the other objects about which we have to count.After that time,the situation begins to be a little less concrete:we use designs(such as pictograms,on walls,stones or paper...)more and more schematized(from a figurative representation of a sheep,to a dash or a point for instance).In this case,the action of representating constitutes exactly the one-to-one correspondance between original objects and their representations.

Then,we begin to regroup sticks or dots and choice a sign (or even a material object such as balls of another rank for abacus) to represent now not a first-order object but a set of these objects to count.With the problem of numeration,of numbers'writing,we arrive to the representation of numbers as cardinals of sets,completely omitting the elements of the sets (in ancient textbooks,there is the distinction "abstract numbers" versus "concrete numbers") and so we begin to represent mathematical concepts instead of material things,and later on it will not be nonsense to have symbols even to represent the number zero or the empty set.

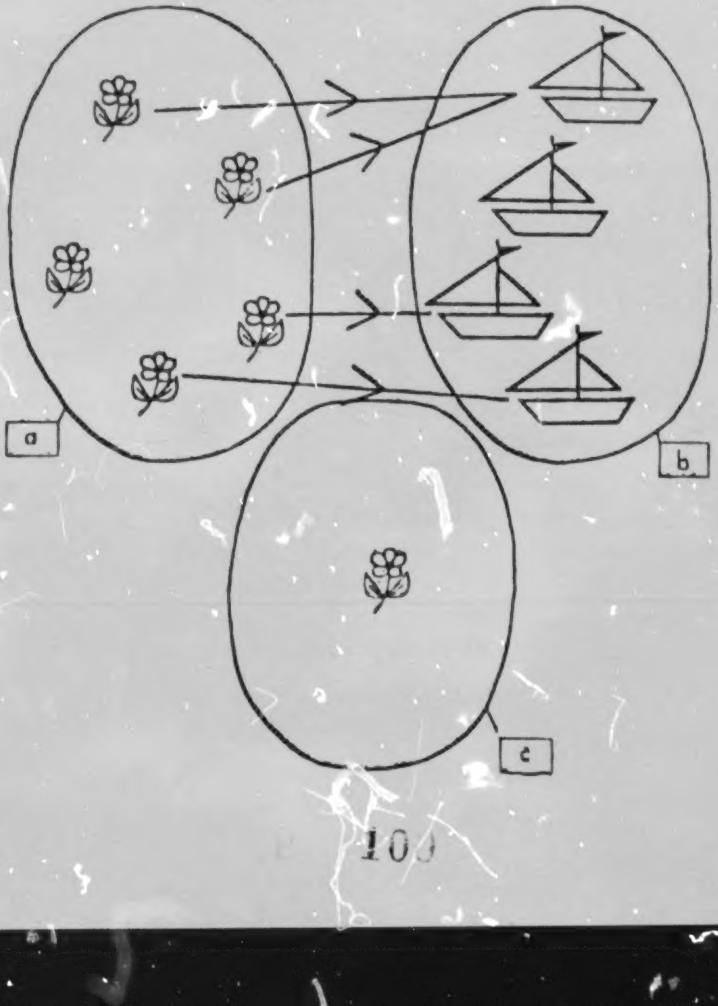
We are also initiated to numerical operations through material and iconic representations of the original objects,for instance addition by joining packets and multiplication by rectangular disposition in lines and columns. But, at the second level, we use a symbolic representation for the numbers and also for the operation; we go from ● ● ● / ● ● ● ● or $\underbrace{|||}\,\underbrace{||||}$ to 3+4 .

When we have to represent equality, symbols are surely not able to be considered as substituting objects themselves, specially because they have to be seen on each side of the sign "=" . So sparkles the perfect absurdity of some "modern-maths" representations of the following two copies from textbooks:

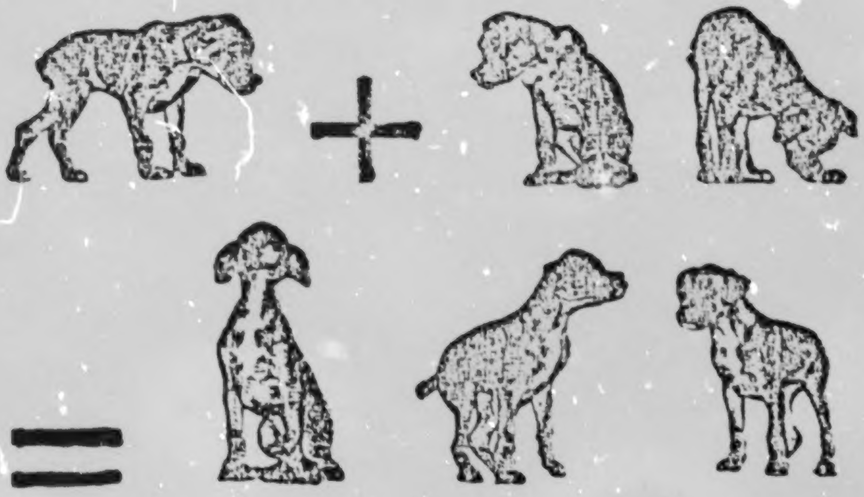
P

Leçon XVIII, Fiche 75

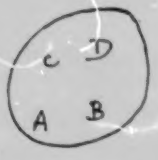
On appelle " ensemble de définition " d'une fonction l'ensemble des éléments de l'ensemble de départ qui ont une image par cette fonction. Voici une fonction d'un ensemble a vers un ensemble b. Complète l'ensemble c pour qu'il représente l'ensemble de définition de cette fonction.



quoted by S. BARUK in " Fabrice" (Seuil)



Supposing that it would be interesting(I am not sure of it!) to represent a set of boats,it would be better to associate a name to each of them(for instance:A,B,C,D)and use the names in the representation (in order to distinguish each of them). But what aboutthis:



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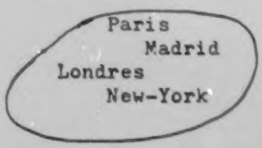
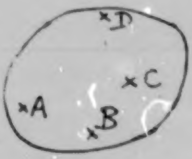
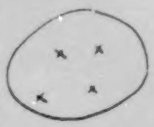
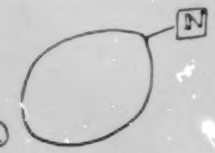


The Venn diagram is intended to realize representation by means of thepoints of the plane,so it seems necessary to mark them like this:

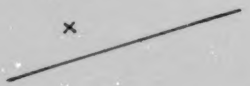


Having decided these coherent directions for use, we see immediately contradictions with others traditional representations (cF[4]), for instance ① for a set of four points without naming them or ② for the infinite set N

①


②


Dealing with the points of the plane, we come to what is, in my opinion, the most complex situation, here. Everybody knows the MAGRITTE's picture "ceci n'est pas une pipe" representing a pipe. Surrealists are not alone to stress the distinction between signified and signifier and its importance for mathematical symbolism is well-known [3]. As Wacek ZAWADOWSKI does in his mathematical textbooks, it is very essential to write, beside representations of a cat and of a triangle: "this is not a cat, this is not a triangle", for if everybody understands that the picture of a cat will not mew, on the contrary, nobody knows a true triangle. Surely, in ④ the

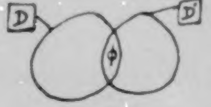
④


inktrace is too thick to be a point, too short to be a straightline but the most essential fact is that the mathematical points are not even atoms of inkblot and the mathematical plane is neither blackboard nor paper; they only stand in brains. Many experiments with children are now done in this direction but they often produce confusion (see analysis of the difficulties inherent to these explanations in [1]).

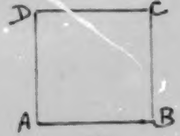
For 3-dimensional geometry, the risk of confusion seems to be less important but the 2-dimensional representation can be intended to be a"photo of the mathematical object"(actually,contemporary textbooks are full of photos).

However,though we stress a straight line on the black-board is not a mathematical straight line,we do not consider as healthy the representation of two parallels by ⑤ !

⑤



Mathematical representations have to be means for visualization of abstract objects and the best representation is generally the one with which most properties can be seen.Sometimes,however,we choice the one which is the best for showing one particular property. Consider a square ABCD,representation ⑥ is the most frequent; ⑦ would be admitted when ABCD is the basis of some cube for instance, but in many other cases people will not admit it.Prejudices are so established that ⑧ would be called only "rhombus".The little ⑨ could be said to "be" the enclosure of a square garden of 50 meters by side.



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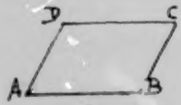

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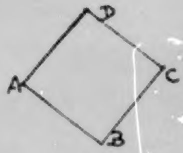

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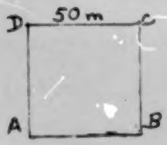
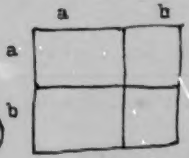


⑨

Actually,more than this geographical mapping role,the geometrical representation has to be an aid for demonstration and then we know that geometry is "the art of right reasoning with false figures". For instance ⑩ even with bad design (and seen through lens) is excellent to express $(a + b)^2$.



⑩

So,for graphical representations,we have to distinguish the use for a support to reasonings through mental images versus the quite physical use of a material object as the ancient "proof" about equal triangles by tracing-paper slipping upon drawn triangles. The confusion between graphical representations and their signified is frequent with arrows diagrams;more and more people forget relations and treat "modern-maths" as study of arrows,and math-questionings are frequently only questionings about drawings more than about mathematical concepts [2].

All"non verbal communication devices"(expression by F.LOWENTHAL) raise this problem:for instance,material representation,as"logic blocks" are often used in a quite perverse manner [1] : to construct concrete representation of Venn diagrams and to call it "set"! And in the ultimate level,we arrive to the "dressing" representation by way of word-problems:for instance,the representationof the addition of 3 and 4 with the story of a boy receiving 3 candies from his mother and 4 candies from his father.I have already written for the PME-group about various parasites created by these school-dressings;here I want to compare this representative relationto the one of the beginning ofthis paper:they are quite inverse one [illegible] the other.Are theytogether mathematical representations? I dare to say that only the first one is"mathematical representation"while the second one is "school-maths representation".We often see the two directions:for instance,when the teacher asks to Peter:"Jane is 4 years old,his brother is 3 years older than Jane;how old is he?" and,if Peter answers "4+3=6",the teacher can say:"Peter,you have 4 marbles and I give you 3 more,how many marbles have you?"(without giving him any marble!)

So two questions will arise:"if A is a representation of B,is B a representation of A?","is the representation of a representation of A,always a representation of A?".I want to say "no" to these two questions such as to "is A a representation of A ?"

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PAST EXPERIENCE AND MATHEMATICAL PROBLEM SOLVING

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This paper examines the relationships of some cognitive abilities, some personality traits and attitudes to children's performance in mathematical problem solving tasks. The discussion focusses on the role of past experience, or long term memory, in the development of some cognitive abilities and behaviours characteristic of some personality traits. A. R. Luria's model of problem solving is discussed with reference to recent Western research based on that author's model of brain function and the effects of differing school experiences on the problem solving process.

Recent research on problem solving in the field of mathematics education has been most notable for its diversity and inconclusiveness. One senses an extreme tentativeness about where education is and where psychology is with regard to mathematical problem solving. Educational papers describing research into teaching procedures appear inconsistent only in the conclusion that *experience* is a key factor in the development of ability. Psychologists, on the other hand, following the considerable influence of Piaget (1970) on early instruction in mathematics have focussed their attention on the assessment of specific *cognitive abilities* or traits and their value as predictors of achievement in mathematical problem solving tasks.

Thus the neo-Piagetian researchers such as Pascual-Leone (1970) and Case (1978) have reinterpreted Piaget's descriptions of developmental stages in terms of measurable quantitive increases in the *capacity* of subject "M -space" or "working memory". The model has been less than satisfactory as the basis for educational procedures partly because "M -space" is somewhat effected by personality and affective variables (e.g. - anxiety Eysenck, 1976) and partly because although information processing is limited at each stage, effective use of "working memory" appears to depend at least in part on *learned strategies*

Mathematics educators have expressed interest in the role of visualization and visual/spatial information processing in mathematical problem solving. Again, most research has been directed at establishing the value of information processing abilities or traits as predictors of success in mathematical tasks, (e.g. McFarlane-Smith, 1972; Krutetskii, 1976; Bishop, 1972). In marked contrast to other research in this field, Ivanova (1980) discusses the *processes* involved in developing generalised visual schema as aids to mathematical problem solving.

The two modes of information processing proposed by A.R. Luria (1973) have received some attention by western researchers. Research by Das (1972), Das and Molloy (1975), Kirby and Das (1978) and others, has confirmed the presence of two distinct factors in information prcessing among normal school populations. These may be hypothesised to represent the separate contributions of *simultaneous* and *successive* synthesis/analysis as described by Luria. A study by Hunt, Randawa and Fitzgerald (1976) suggests that successive processing may be associated with rote learning and simultaneous processing with higher order thinking skills associated with the comprehension and analysis of relationships. Studies by Green (1979) and Lawson and Kirby (1981) suggest that *past experience* (e.g. teaching procedures) is influential in the mode of information processing used. Research by the author[1] suggests that although both modes of processing are available to five year olds *simultaneous* processing is not a potent factor in *early school assessment tasks in mathematics*. Western research based on Luria's model of brain function have also confirmed the apresence of an 'attention' or 'planning' factor which may be hypothesised to represent the contribution of the functions of the frontal ideas of the cortex as described by Luria.

Until recently, little attention has been paid to the affective aspects of mathematics education. However a spate of descriptive literature (e.g. Tobias and Weisbrod, 1980) has discussed possible relationships between anxiety, field independence, inpulsivity and other personality *traits*, and achievement in mathematics. These studies have in general done little to explain the phenomenon of 'mathphobia' among otherwise confident and competent individuals. The description by Davis (1972) of egalitarian teaching procedures in classrooms where mathematical problem solving apparently flourishes, is of note. Also the work of

Block (1980) and Messer (1976) suggests that the personality continuum from impulsivity to reflectivity are important not only as factors in goal directed behaviour such as problem solving but also in the process of the development of conceptual schema. In particular Messer suggests that reflectivity requires the active involvement of the subject in the decision making associated with the task at hand. The work of Skowronski and Carlston (1982) suggests that *desire* for such participation and personal control is the result of positive outcomes from *past choices* In general the social learning theories of personality as described by Wright and Mischel (1982), Alloy and Selligman (1979) and Staats and Burns (1982) appear to provide a more adequate explanation of some aspects of the complex process of mathematical problem solving within the school context. A recent study by Skon et al (1981) suggests that social learning, in the form of co-operative peer interaction experiences, is *also* important for the development of higher order reasoning strategies among young children.

The discussion so far has emphasised the function of past experience (and by inference long term memory) as an important factor in the cognitive processes and the affective behaviours exhibited by children solving mathematical problems.

The author's current research project is an attempt to further explore the problem solving behaviours of children at the upper primary school level. To this end two data collecting procedures are being used with a group of ten-year-old children. Firstly, individual interviews are conducted using Luria's (1973) model of problem solving as a framework and a co-operative interview technique first developed by Ransley (1979). By using Ransley's method it is possible to collect data on problem solving behaviour at all stages of the process rather than discontinuing the interview at the first cause of difficulty. Problems are chosen so that task demands are within the expected mathematical competence of this age group but structured so that reasoned decision making rather than overlearned strategies are required to achieve a solution.

Secondly, a questionnaire designed to determine attitudes and salient beliefs about the following :

- the *value* of estimation as part of the problem solving process;
- the consequences of such behaviour;

- the normative pressures from significant others concerning this behaviour;
- self evaluation of motivation to comply with such pressures.

The selection of *estimation* as a target behaviour for this survey resulted from clinical experience which suggests it as an indicator of a reasoned participatory approach to problem solving tasks. The questionnaire has been constructed according to the procedures suggested by Ajzcen and Fishbein (1980) in their theory of reasoned action. At the time of writing, the results are incomplete but tend to confirm the ideas expressed in the early parts of this paper. In particular, problem solving profiles resulting from the individual interviews appear to reflect student *experience* in mathematical problem solving *as well as* ability differences in attention and information processing. Classroom observations and previous research by the author[2] (1982) suggest that in addition to differences in experiences caused by variations in curriculum emphasis and teaching style from classroom to classroom, *most* teachers intervene in the problem solving process to 'help' those students who are perceived to be experiencing difficulties, and expect independence from capable students. Each intervention reduces the experience of the former students in the processes of decision making associated with the problem solving process, and as a result reduces their opportunity to develop effective strategies, take risks, make choices, analyse relationships and evaluate outcomes.

Notes 1 & 2 : the author has previously published as K.P. Grabham

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MATHEMATICS AND SELF-ACTUALISATION

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I. The TM-Programme and Self-actualisation

In the urge to realize his inner potential, man experiences desires, engages in action to fulfil those desires, accomplishes his desired goal, and finds his personnality enriched with the confirmation or reinforcement of a new aspect of his potential. From the attained state of satisfaction new desires arise and the process repeats itself, not merely multiplying experiences on the same level, but tending to ascent in intentionality and extentionality. In this way man explores or constructs the hierarchy of successively generalised structures of his potential. In order that each newly opened aspect of the personnality is actualised, that is, made operational and leading to fulfilment of desires, one's actions have to take into account a greater set of laws of nature at each stage. So the growth in self-actualisation leans on an increased ability to function in accord with more generalized levels of natural law.

The human nervous system is a sophisticated instrument which is capable of reflecting on its own processes and to intuit from them the functioning of the laws of nature. This perception of the laws of nature is the basis for one's attunement to them in one's actions, which in turn is at the basis of reaching the desired achievments.

The main blockages hindering an optimal friction-free passage to higher states of self-actualisation are stresses accumulated in the nervous system due to overtension. Like impurities in a mirror telescope they reduce the refinement of discrimination, so that the attunement to structures of natural laws is arrested at a grosser surface level.

The Transcendental Meditation-technique (TM) is a mental technique in which the mind transcends the perceptive threshold of this surface level and experiences its activites in their more refined, general stages. On a physiological level this experience is accompanied by a deep state of rest - deeper than sleep - while wakefulness is maintained or even enhanced. In this deep state of rest the nervous system frees itself from its material and structural abnormalities (stresses), thereby enabling permanent establishment of the increased refinement in perception. Regular practice of the technique (2 x 20 mns a day) results in a progressive - and at each stage established - refinement in perception, leading one's actions more in accord with generalised structures of natural laws. This results in one's daily life in an easier accomplishment of one's goals, in less time with less energy consumed and fewer stresses accumulated.

More than 700 researches in physiological, psychological and sociological areas have confirmed that this growth in self-actualisation is real, physiologically integrated, that it is continued steadily over the months and years and that it involves all aspects of the personnality. One of the results which researches have pointed out and which speaks for the reduction of stress and friction in one's daily activity is a significant reversal of the ageing process

II. The Nature of the Mathematical Faculty

Amongst psychologers of mathematical thinking the understanding seems now quite commonly established that mathematics arises from a kind of reflection on our interiorised sensori-motor activity. As this interiorisation runs over a rather continuous scale from material to increasingly mental, we find that mathematics thematises the structures underlying our thinking processes as well. This is illustrated by the fact that lower-level processes of mathematical activity are reified, i.e. made into objects of study at higher levels.

The basic capacity involved in this I would like to evoke with the following quotes :

> "The dive of the eagle on his prey, the spring of the tiger - demand calculation. Differential and integral calculus, geometrical functions, even though no knowledge of geometry."
>
> "The work of an acting man is a magnificent condensation performance; the condensed result of a vast quantity of separate calculations and considerations; the mathematician is a man who has a fine capacity for self-observation of this condensation process"
>
> "The mathematician appears to have a fine self-observation for the metapsychic (also probably physical) processes and finds formulas for the operation in the mind of the condensation and separation functions, projects them, however into the external world, and believes that he has learnt through external experience."
>
> (S. Ferenczi, quotes taken from The process of Learning Mathematics, ed. L.R. Chapman, 1972, Oxford)
>
> "An essential element is still lacking in the dialectic synthesis under consideration : it is an explanation, however vague still, of the faculty that the mathematical person possesses to produce and link amongst each other mathematical forms for informational purpose. An explanation could be suggested by the picture that modern biology reveals about ourselves day by day. It could be that the mathematical person is, up to the least of his cells and parts, an informed being of the means which - here I take up an expression by Jacques Monod - the execution of his project to exist recquires. The concention and mastery of mathematical entities is in the same line as the production and practice of a communication language which itself in in the same line as the genesis and putting to function of the sensorial figurations etc.

> In this vision on things, the mathematical intuition, even when exercised on an abstract level, presents itself as maybe the most evolved form of the necessary contact that man and his species must undertake with the world of their existence."
>
> (F. Gonseth in Actes du Colloque sur Les Mathématiques et la Realité, partly taken from "Dialectica", Revue internationale de philosophie de la connaissance Vol. 29 N° 1 (1975), Bienne (Suisse)).

This conception of the mathematical capacity considered in relation to the mechanics of self-actualisation described in the first chapter put before us some questions to be researched and possibilities for mathematics education to be considered. We will look more closely into the following points : (III,IV)

III. Parallels between the Enhanced Process of Self-actualisation and the Processes of Mathematical Thought

For want of space I will here not enter into the mathematical activities on micro-level, that is to say the mathematical mental actions like conjecturing, proving, defining ... . Their coördinated functioning forms networks which on their own level undergo processes obeying certain regularities. It is on this latter level that we see hierarchies of structures appearing, the characteristics of which reflect most appropriately the evolution in self-actualisation. We will now take a closer look at the evolution through this hierarchy of structures. Every step or characteristic mentioned can be interpreted as a mathematical move on the formal level or a self-actualising move on the operational level.

Most strikingly in common between mathematical activity and the process during the TM-technique is that while the attention is focused on the topic under consideration, certain meta-processes governing the coördinated functioning of the mental actions guide the mind towards an increasing freedom from the contingent characteristics of the concepts in the topic. The mind thus liberated from its limited interpretations but remaining focused on the essential information in the topic, is left to its own internal dynamics and reconstructs this information level after level.

Each structure of knowledge in this chain transcends the previous one whilst reconstructing and integrating it. The transition from one level to the next has an aspect of continuity (in the transformatory process) and discreteness (newness of the attained structure). The dialectic between one's experimental intuition on the starting level and the new structure in formation modulates on this continuous transformation. The process of verification with experimental intuition is gradually replaced by intrinsic moves in the new structure, and one's reference foundation gradually shifts onto the new structure.

The previous level gets reconstructed, its elements are now seen in the wider framework of the next level which unites more of the diversities of the former, and the span of which encompasses a wider range of application. One looks at the world rather through its underlying transformation processes than its objects, one lives more in the derivative. Problems of the starting level are solved or clarified, and most often one gets the solution for a whole bunch of problems all at once. One leaves problems behind in levels. Much energy is saved by moving directly within the new structure, and the particularisation,onto the starting level, of the results there obtained, is automatised.

The new structure tends to increase its internal coherence, unnecessary loops are purified out, paths of relation are curtailed (stress release), it grows into a more simple, coördinated whole. This structure of knowledge obtained expresses more of the essence of the topic under consideration. The apparent move away from the real world has led to an inexplicable more evolved contact with the world (well-known in the case of mathematics; parallelling this there is an inexplicable increase in the appropriateness of one's spontaneous actions in daily activity with regular practice of the TM-programme).

As the new structure approaches a certain closure, its autonomisation is accelerated, its mental moves are reified, i.e. they are made into objects of attention in their turn. Reflection shifts on to uncover the structures of their coördination and the whole cycle starts again.

Basic faculties in this process are the self-reflective ability and a sense of esthetic sensibility that works as a directional force in the move more and more autonomously away from real-world situations. This move is not arbitrary, its meaningfulness being nourished from a "regular return to the sources" (R. Thom), these sources being of two kinds : new elements of perception either of the outside world situation or of internal mental actions used unconsciously till then.

IV. Mutual Fertilisations of Mathematics and the TM-programme : Promisses for a Coördinated Growth in Self-actualisation

a. Enhancing mathematical faculties through the TM-technique

In order to envision a bi-directional fertilisation between mathematical activity and the process of self-actualisation, we have to point out their commonality, their difference and their genetic relatedness. The main idea is that we have to do with parallel processes, occuring on different levels however : mathematical activity operates on the contents of man's awareness whereas the TM-technique operates directly on the substance of awareness. The relation between these two is that the latter is prior to the former in genesis. That is, through the TM-technique a man starts acting from a more universal basis in his awareness, whether or not he is able to thematise its structure. With the release of stress this more universal aspect of his inner nature is opened up to become operational. Awakening these layers of potential one by one prepares for a later thematisation of their structure, or the recognition of such a thematisation when presented with in education. Here lies a major possibility for contribution of the TM-technique in mathematics education.

Moreover, as is clear from chapter I, the TM-technique directly develops the self-reflective ability of the mind. I would like to limit myself here to pointing out a relation between reflective intelligence and esthetic sensibility. In the process of witnessing one's own mental activities on increasingly primordial levels of their emergence during the TM-technique, one comes to a situation where all mental activity is transcended and pure self-reflectivity remains, i.e. one is alert, awake in a state of pure self-consciousness, unaccompanied by mental activity (no thoughts). Now as this experience is one of a unified state - all diversified experiences reduced back to their single, common source - it follows that every act of reference to this state is an act of esthetic verification. One refers a multitude of diverse experiences back to a coherent whole. Through regular practice the experience of self-awareness becomes established along with the experience of mental activity, thereby spontaneously estheticising all thoughts and actions.

There are other results of interest for mathematics education (increased field independance, increased ability for abstraction, memory shift towards generalisation mechanisms), but in this summary we restrict attention to the basic ones.

b. A new role for mathematics education : inform man about the successive structures of his growth in self-actualisation

It is important to remark here that the TM-technique establishes these effects directly on an operational level. We can however imagine that thematisations of the structures underlying our evolution, introduced at the right moment, could reinforce the operational knowledge already functional. Here we can envision a new role for mathematics education : mathematics could assume the role of a reflector, mirroring stage after stage man's structured evolution to higher states of equilibrium.

For want of space we will have to limit ourselves here to one example of such a reflection, in the area of Galois'theory. In the history of algebra we see how the community of mathematicians, in its endeavour to solve equations, has been led through successive extensions of the number concept. Confronted with a fragmentation crisis at the end of the 19th century, algebraists were forced to reflect upon their own mental functionings in solving equations, and have brought out their coördinating structures. The resulting Galois' theory teaches us how to systematically reduce the group of an equation, and with the information gained from that, expand the field of the equation until it encompasses the roots. For a practitioner of TM this offers an exact description of his daily experience of solving problems (operationally) through the expansion of his awareness. The unattainability of a solution is due to the entropy in the metabolic functioning of his nervous system, pushing one's awareness into the more excited, narrow surface levels of one's mind. Through a systematic reduction of the metabolic level, consciousness is allowed to expand step by step until the solution level of the problem is at hand. Regular practice accustoms one to the different field extension levels of one's awareness and their corresponding metabolic states. One thus obtains an "operational feel" for the main theorem of Galois' theory. In due time, as the perception of our mental processes will refine further, finer details of Galois' theory will find their meaningful counterpart in one or other aspect of our problem-solving psychology.

There are other examples, the most interesting of which relate man's experience of the foundations of his awareness in higher states of self-actualisation to the foundations of mathematics and algebraic closures.

"REAL-LIFE" NUMERACY, ARITHMETICAL COMPETENCE AND PREDICTION FROM A NEUROPSYCHOLOGICAL THEORY

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The work of the Russian neuropsychologist, A.R. Luria, has been attracting attention from western researchers in many psychological fields. Luria's theory of brain functioning, arising out of extensive work with brain damaged patients, has increasingly gained acceptance as having important significance in studies of individual differences.

To date, attempts to use Luria's theory as a basis for studies of individual differences have focussed on a particular suggestion he made in 1949 (Luria, 1966, p.577). At that time, Luria postulated the existence of two general categories of information synthesis which he suggested as a starting point for investigating the factors of mental processes. The first was concerned with integrating elements into simultaneous groups; the second with the synthesis of successive, serially constructed processes, or the combination of elements into series. Psychological analysis of disturbances of mental processes in patients with local brain lesions revealed that some sections of the brain are responsible for forming simultaneous, while other sections are responsible for successive syntheses. The parieto-occipital areas are responsible for the simultaneous, while the temporo-frontal regions contribute to the formation of successive synthesis.

Several western researchers have taken up this suggestion of Luria and have attempted to operationalize the simultaneous/successive (sim/succ) constructs with psychometric test batteries. Factor analytic techniques have consistently yielded factors labelled as simultaneous and successive processing. Several investigators have had success in applying the sim/succ psychometric model to educational settings, notably in the area of Aptitude treatment interaction (ATI) studies. Green (1977), Grabham (1980) and Walton (1983) each demonstrated the existence of interaction effects between aptitudes in sim/succ processing and differing instructional treatments in school mathematics.

Ransley (1981) working with 5 and 6 year old children extended a simple sim/succ model to encompass a third factor. The new factor was labelled "sustained selective attention", and its existence was predicted from Luria's work with patients suffering "frontal" damage to the brain. The three factor model was found to be educationally relevant by demonstrating considerable power in accounting for individual differences in classroom learning. The competency of the children was measured with a representative range of classroom skills determined by teacher assessment, the application of standardised achievement tests, and the use of experimenter constructed achievement tests. Factor scores were generated from the three factor solution in order to define three variables corresponding to simultaneous synthesis, successive synthesis and selective attention. These variables were used as predictor variables in relationship with sets of measures taken from the classroom competence battery. The use of canonical correlation analysis and multivariate multiple regression analysis supported the contention of highly predictive contributions made independently by each of the simultaneous, successive and attention variables (e.g. canonical correlation coefficients as high as 0.74 were obtained).

Given the success of the model in prediction of school achievement for young children, it is natural to enquire if the same success would be apparent with older children. In particular, the effectiveness of the model for explanation of differences in mathematics performance could be of interest. In Australia, as in most other countries, the alleged poor numeracy skills of many school leavers has been of recent concern in our community. Psychometric models based on Luria's neuropsychology hold potential for explanation of mathematical difficulties. As a first step, investigations are needed to test the effectiveness of a model to predict individual differences in performance among those experiencing difficulties. The remainder of this paper reports a pilot study which attempts to explore this question and to reveal something of the nature of the difficulties experienced by many pupils in the numeracy area.

SELECTION OF SAMPLE

A sample of 122 fifteen year old children was selected from among those who scored poorly on a standardized basic numeracy test (14N Test) administered by the Tasmanian Education Department one year ealier. The pupils were drawn from 7 High Schools in the Hobart city region. The mean for the sample was 11.72 out of a possible total of 33. S.D. = 3.38.

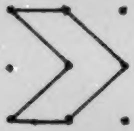
TESTS FOR PSYCHOMETRIC MODEL

In the 1981 study the model was initially developed with the aid of a microprocessor and other electronic hardware in attempts to parallel the work of Luria. Such techniques are not readily adaptable for ordinary classroom use and so a second set of tests was devised requiring little specialised equipment. The second set of tests are readily handled by teachers, counsellors and other school personnel who are concerned with individual differences in young children. The measures obtained with the more complex measures were used to validate those obtained more simply. In fact, it was demonstrated that for everyday school use a practitioner could obtain rough estimates of the abilities by administering only one test from each domain. Consequently in the present study three of the tests were administered and the appropriate raw scores were taken as approximations to the relevant abilities. The Dot Figure Memory Test (DFMT) was used for simultaneous synthesis, the Symbol Tapping Test (STT) for successive synthesis and a Cancellation Test (CT) for the attention component. The procedures developed for administration of the tests with the younger children were found to be suitable for use with the older sample.

Dot Figure Memory Test (DFMT)

The test apparatus consists of a set of drawings on 3 x 3 dot grids.

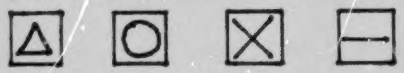
Example:



The child is shown each in turn (10 secs) and required to draw them on blank grids from memory. In terms of Luria's theory this test should measure simultaneous synthesis at the perceptual/memory level, particularly involving the visual sphere.

Symbol Tapping Test (STT)

In this test children are shown symbols one at a time on cards. They are required to indicate by tapping an answer strip the order in which the symbols were presented. Marks are awarded for each symbol in the correct position. Four different symbols are employed:



Test items contain between 2 and 5 symbol sequences.

Luria's theory suggests the test should measure successive synthesis at the memory level.

Cancellation Test (CT)

The test sheet consists of a matrix using four different shapes ⊏ ⊔ ⊐ ⊓ The child is required to place a dot in the centre of each of the shapes identical to the target ⊏. The subject has to start at the top left hand corner and work left to right along each row in turn. They must work as quickly and accurately as possible. The 1981 study indicated that the total time taken to complete the test is a good rough measure of attention ability. The ability to differentiate items on the matrix rapidly and accurately involves focussed selective attention.

NUMERACY MEASURES

"Real-Life" Numeracy

During work concerned with diagnosing difficulties experienced by children in solving mathematical problems (Ransley, 1979) it was noted that, for many children, skill in mechanical computation (symbolic numeracy) seems to be independent of understanding and application of arithmetical operations in "real-life" settings. Similar observations have been reported from many other sources (e.g. N.C.T.M., 1977). In mathematics education circles one can discern a sense of agreement that for many children the traditional group administered tests are inappropriate for assessing numeracy levels. Many of the skills and understandings appropriate to younger children (and older ones experiencing difficulties) cannot be adequately assessed in the paper and pencil format. There is a growing realization that if "numeracy" is to be fairly assessed a means must be found to measure performance in more realistic formats. In this study "Real-life" numeracy was measured by items on the Practical Understanding and Application of Concepts test (PUAC). The 10 items are administered individually and attempt to tap practical understanding of concepts, ability to apply knowledge, and power of explanation of processes. The items require the use of accompanying concrete materials and other stimulus items, with a minimum of symbolic representation, in order to simulate "real-life" settings. Two items relating to fractions are shown:

3. Requires a jar with 45 counters in it.
   Give them the jar and ask: *Can you find two thirds of these for me?"* *"Tell me how you did it."*

6. Use two identical cut out circles (r=5cm) side by side. Ask:

"These are pizzas, which would you rather have; one fifth of this pizza (point) *or two tenths of that pizza* (point)*?"*

"Why?" (Allow them to explain - you may need to ask additional questions e.g. *"Would you have more to eat?"*). Marked incorrect if they do not believe equal amounts would be had to eat.

Symbolic Numeracy

This was measured by a Mechanical Arithmetic Test (MAT) also consisting of 10 items. Each item matches one in the PUAC test. The items matching the examples are:

"Work out each answer as far as it will go."

3. $\frac{2}{3} \times \frac{45}{1} =$

6. $\frac{5}{10} =$

Matching the two tests allows for possible interesting exploration of the relationship between "real-life" measures and the more usual symbolic approach.

Test Administration

All testing was administered by two Senior Graduate Research Assistants who were thoroughly trained to the satisfaction of the researcher. Testing was spread over two individual sessions, during October 1982, with the Mechanical Arithmetic Test being given last.

Analysis of Results

The comparison of items between PUAC and MAT could make a fascinating study on its own account. Cross tabulation for the two examples are shown.

3.

		PUAC		
		Right	Wrong	
MAT	Right	16	28	36.1%
	Wrong	10	68	63.9%
		21.3%	78.7%	

6.

		PUAC		
		Right	Wrong	
MAT	Right	26	34	49.2%
	Wrong	27	35	50.8%
		43.4%	56.6%	

The pupils were significantly better at straight mechanical performances than the corresponding applications. MAT mean = 6.90, PUAC mean = 6.32, Related T-value is 3.22, df = 121, p = .002. The result is perhaps indicative of a lack of attention given to the understanding and "real-life" aspects of basic arithmetic in our schools. The results on both tests are disturbing, given that the majority of pupils left school within weeks of testing.

Taking scores on the three psychometric tests as a set of predictors and the scores on the 14N Test (after a logarithmic transformation) along with the numeracy tests as criteria variables, a canonical correlation analysis was performed using the Multivariance VI program (Finn, 1977).

The analysis revealed the existence of one highly significant canonical correlation coefficient = .60, chi-square = 64.30, d.f. = 9, p < .0001. The standardised weights and correlation with the canonical variate are shown for each variable.

Predictor Set	Weights	Correlations	Criterion Set	Weights	Correlations
Figure Matrix	.67	.38	14N Test	.91	.99
Symbol Tapping	.16	-.41	MAT	.02	.59
Cancellation	-1.08	-.75	PUAC	.17	.53

Inspection of the correlations for the criterion set indicates that each test correlates highly with the shared underlying dimension of variation. The nature of this dimension can justifiably be described as one involving the assessment of basic numeracy. The correlations of the predictor set variables with this dimension indicates that the variables measuring simultaneous synthesis, successive synthesis and selective attention all contribute significantly to the prediction. Supporting evidence was provided by a multivariate multiple regression analysis, successively adding the predictor variables to the prediction of the set of criterion variables, each addition being highly significant.

Given the restricted range of ability chosen on the 14N Test the results are an underestimate of the predictive power for the total population of 15 year olds. As well, the psychometric tests are at best only rough estimates of the abilities in the Luria model. The findings are encouraging and suggest that a full study could give impressive results indeed.

Luria's theory of brain functioning has led to the development of a simple psychometric model with obvious relevance to the classroom. Measurement of competence in the model can be carried out using tasks which do not require any special formal knowledge and do not demand any particular higher order intellectual functioning. The technique described seems to provide a means for directing research into the basic components of mental processing. Methods for remediating numeracy skills which are designed to interact with these basic abilities, should prove useful in helping children overcome their difficulties.

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LEARNING OF NON-STANDARD ARITHMETIC AND THE HEMISPHERES OF THE BRAIN

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1. INTRODUCTION

The relation of ordinal and cardinal mathematical concepts to the left and right cerebral hemispheres, respectively, has been examined in Fidelman (1). The method of research was to find statistical correlations between scores on ordinal and cardinal mathematical concepts and hemispheric tests.

The hemispheric tests that were applied are:

1) Counting of tachistoscopically represented dots as a test for the right hemisphere.

2) Counting of signs appearing rapidly one after another as a test for the left hemisphere.

2. NON-STANDARD ANALYSIS

The students of first year mathemtics at the Technion, Haifa, Israel, learned calculus in two parallel courses. One course was the standard approach, and the other was based on the non-standard approach. The text book in the n. s. approach was Keisler (2). The students participated in the hemispheric tests.

Results of this experiment given in (1) are in line with the conjecture that potential infinity is related to the left hemisphere, while actual infinity is related to the right.

Additional results of this experiment are given in table 1. They show that understanding of of the algebraic structure of the n.s. real numbers (monads and galaxies) may be related to an inhibition of the left hemisphere by the right hemisphere.

We may explain this phenomena by the following conjecture: The cognitive concept of infinity is first introduced by the left hemisphere (which processes discrete details of data one after another) as potential infinity. The right hemisphere (which creates a new whole from discrete details of data) integrates actual infinite sets, from the elements of series created by infinite potential processes, at a later stage. This integration terminates a process without a last step which is created by the left hemisphere. It is accomlished by an inhibition of the left hemisphere by the right.

Table 1: Correlations between scores on the algebraic part of n.s. calculus and neuropsychological tests (n=11)

Neuropsychological counting test	rho
1) Simultaneous counting of dots by true/false	.657 ** (one-tailed)
2) Average of all ordinal counting by true/false	-.556 * (two-tailed)
3) Hemispheric dominance: Difference between (1) and (2)	.851 *** (two-tailed)

3. PARADOXES AND DIAGONAL PROCESSES

Another experiment surveyed in (1) supports the conjecture that learning of diagonal processes and set theoretical paradoxes are related to the inhibition of the right hemisphere by the left. We sugested that this inhibition is part of a cognitive process in which the right hemisphere integrates the set of ALL elements having a certain property. The left hemisphere creates an additional element having the same property. Thus the left hemisphere 'breaks' the set which the right hemisphere has integrated. The left hemisphere must overcome the 'objection' of the right hemisphere to the 'breaking' of the set of all elements having the certain property.

Actual infinity is not created by diagonal processes. But once the property of being actually infinite set is defined and not empty, more elements having this property are created by diagonal processes.

4. NON STANDARD ARITHMETIC

Five experiments have been conducted at the Technion in which students of a course on philosophy of mathematics learned compactness theorems proving the consistency of the existence of one of the following: an infinite natural number, an infinite real number, an infinitisimal. In 3 of the experiments the students leaned also about internal and external sets in n.s. models of arithmetic. The examinations included a question on the compactness theorem. When the students learned also about internal and external sets, the examination included a question on this topic. The students participated in the hemispheric tests. The scores of the mathematical and hemispheric tests were correlated.

In four of the experiments, including the 3 with learning of external and internal sets, the students did not learn about monads and galaxies. The results of the experiment having the largest sample among them are in tables 2 and 3. The results of the other three experiments are similar, but less significant.

Table 2: Correlations between scores on the existence of infinite natural number and neuropsychological tests (n=29)

Neuropsychological counting test	rho
1) Simultaneous counting of dots by true/false	-.152 n.s. (two-tailed)
2) Average of all ordinal counting by true/false	.381 ** (two-tailed)
3) Hemispheric dominance: Difference between (1) and (2)	-.398 ** (two-tailed)

Table 3: Correlations between scores on internal and external sets in n.s. model of Peano and neuropsychological tests (n=29)

Neuropsychological counting test	rho
1) Simultaneous counting of dots by true/false	-.112 n.s. (two-tailed)
2) Average of all ordinal counting by true/false	.667 *** (two-tailed)
3) Hemispheric dominance: Difference between (1) and (2)	-.503 *** (two-tailed)

In the remaining experiment the students learned the compactness theorem about the existence of an infinitisimal. After it they learned about monads and galaxies (which were not included in the test) but not about internal and external sets. The results are in table 4.

Table 4: Correlations between scores on infinitisimal (with monads & galaxies) and neuropsychological tests (n=17)

Neuropsychological counting test	rho
1) Simultaneous counting of dots by true/false	.686 *** (two-tailed)
2) Average of all ordinal counting by true/false	-.054 n.s. (two-tailed)
3) Hemispheric dominance: Difference between (1) and (2)	.470 * (two-tailed)

5. DISCUSSION

There is similarity between the results for diagonal processes and paradoxes in (1) and the results for the existence of infinite numbers and infinitisimals, when the students did not learn about monads and galaxies (table 2). This infers that the processes involved may be similar. The cognitive process may include integration of the set of all (finite) numbers at first stage by the right hemisphere. At a later stage an additional infinite (or infinitisimal) number of the same set (natural or real number) is extracted by the left hemisphere. Thus the left hemisphere 'breaks' the set of all natural or real numbers which has been created by the right hemisphere. This may be done by an inhibition of the right hemisphere by the left.

The results for internal and external sets are also similar to those of diagonal processes and paradoxes, and may be explained as indicating an inhibition of the right hemisphre by the left. The inhibition may be related to the cognitive process in which the elements of an external set can not be integrated by the right hemisphere into a set of the system, and must be treated as single elements by the left hemisphere.

The results of the experiment in which the students learned about monads and galaxies after the compactness theorem (table [illegible]) are similar to the results for monads and galaxies in the course about n. s. analysis (table 1). The explanation may be that while the students learned the compactness theorem for the first time, it has been related to an inhibition of the right hemisphere by the left. But the learning of galaxies and monads created a mental set theoretical model of the n.s. stpucture of numbers. The creation of this model is related to an inhibition of the left hemisphere by the right because it is actually infinite. While answering the question of the examination, the students used this model to understand the concept of the infinitisimal. The using of this model is related to the inhibition of the left hemisphere by the right.

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PRACTICAL APPLICATIONS OF PSYCHOMATHEMATICS AND NEUROPSYCHOMATHEMATICS

by

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1. INTRODUCTION

Mathematical psychology is the investigation of Ψ by mathematical means, and psychomathematics (or psychological mathematics, ΨM) is the investigation of mathematics (-learning) by psychological means. Expressed otherwise: that approach to mathematics, which takes into account not only the logical connections, but also the human way to reach them. If we include in the meaning of the words "human way" also the study of the connection between brain functioning and behavior, then we deal with neuropsychomathematics (NΨM).

ΨM and NΨM are relatively new branches of research. They are in the juncture of at least four sciences: Ψ, mathematics, physiology and education. For this reason progress is difficult and slow. Nevertheless there are some results. We want to deal here with their practical applications only.

2. PSYCHOMATHEMATICAL TEACHING METHOD (ΨTM)

In case of difficulties in learning to read, Carmon suggests [1]- and practises - to examine which half of the brain of the child is dominant and teaches accordingly; sequentially, letters-syllables-words-sentences, or holistically, ideographically as in Japanese. He even suggested to try the same kind of treatment with children having difficulties in arithmetic, but no research has been done till now. Perhaps this would be possible with elementary school mathematics, where in most of the topics both sequential and global presentation is possible. But in secondary school mathematics each branch of mathematics has its own mode of presentation. Furthermore while elementary school children still are in the development phase of the concepts necessary to mathematical understanding and generally do not have enough linguistic equipment, this is not the case with secondary school students. Accordingly, for use in the ages from 13 years and onwards, the ΨTM has been developed [2]. Its principle can be summarized as adapting the teaching method to the material taught.

Some parts of the mathematics curriculum are conspicuously sequential, i.e., composed of steps, each of them follows from the previous one and the whole set of steps cannot be grasped holistically. Algebraic operations, solution of equations or their systems, investigation of behaviour of functions are examples. In order to adapt the teaching method to the material taught, in sequential material we use algorithmic rules, i.e., we utilize the possibilities of the left half-brain. The algorithmic rule comes instead of formulae. By the way, in most of the cases the algorithmic rules have the properties of Skemp's schemata [3],[4].

Other parts are global in nature, i.e., the whole problem can and should be grasped at once. Examples are topology, geometry, set theory, but also "great formulae", especially in trigonometry, but also in algebra. To adapt the teaching method to such material we use explanatory rules, i.e., we utilize the possibilities of the right half-brain. The explanatory rule may come together with a formula, but anyway each part of it concerns the whole problem.

In the early phase of the YTM we conducted controlled experiments. The differences between the achievements of the experimental and control groups were significant in most of the cases on the .001 level. Now this is a well-based and successfully practised teaching method.

3. THE COGNITIVE METHOD AND TRANSFORMATION GEOMETRY

Difficulties arise in cases, where both halves of the brain are involved and the usual form of teaching compels the student to transmit the solution process many times to and fro between the two halves of the brain. Two obvious examples of such a situation are the mathematical model formation or, in its most elementary form, solution of story problems and problems in geometry in which the supply of a proof is demanded.

In case of model formation the origin of the difficulties is this: the whole problem is perceived by the right brain globally. Then it is transmitted to the left brain for further elaboration step by step, sequentially. During this process each step must be related to the whole problem: the conveyance to and fro many times between the two halves of the brain may cause delay or interference.

The cognitive method [5] solves the problem as follows. The whole problem is grasped by the right half-brain globally and immediately some global decisions are made in the same area. Then the processing is transmitted to the left half-brain, together with the mentioned decisions. Now an algorithmic rule is used and with its aid the problem will be transformed step by step into mathematical form.

Again, in the early times of the cognitive method, controlled experiments revealed significant differences in achievement between the experimental and control groups. Now this is a well established practise. Even technical aids have been published for its use [6]. (By the way the publisher did not inform me about its publication.)

The success achieved with the cognitive method gave impetus to the conjecture that the problem of problem solving in geometry can be solved in a similar way. That similar way we can find in teaching transformation geometry. The problem should be received by the right brain globally. After some global decisions it should be transmitted into the left brain. In the case of teaching transformation geometry from the very beginning of geometry, the left brain will view the problem from the point of view of transformations, motions, step by step. By an appropriate algorithmic rule there is a possibility to write down the motions in rigorous mathematical form.

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C. COGNITIVE STUDIES IN ARITHMETIC

1. *WORD PROBLEMS*
2. *NATURAL NUMBERS*
3. *FRACTIONS AND RATIO*
4. *OPERATIONS*
5. *APPLICATIONS TO TEACHING*

ORDER OF MENTION VS. ORDER OF EVENTS AS DETERMINING FACTORS IN ADDITIVE WORD PROBLEMS: A DEVELOPMENTAL APPROACH

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The experiment was designed to assess the influence of the order of mention of events and numerical data in the text vs. the order of occurrence in real time of the events alluded to in the text. A developmental trend was found in which younger children are more carried away by the order of the text and older ones are more able to recognize the logico-mathematical aspects of the text.

The question addressed in the present research is connected to the general issue of modes of processing written information within a structural text. The specific question addressed in the present research concerns the influence of the order of mention of events in the text as compared to the order of occurrence of the events alluded to in the text, and the order of numerical data. It was undertaken to try and answer that question using as a text a simple additive word problem.

This kind of analysis requires taking into account three different aspects of the text:

a) The logico-mathematical aspect.
b) The "real-life" situation described by the text.
c) The order-of-mention in the text of
  I. the events
  II. the numerical data

An experiment was designed so that the influence of these three aspects could be contrasted.

Procedure

Ss were 100 children of two grade levels (2nd and 6th). The tasks were six verbal arithmetic problems, four of which were of similar semantic structure (change problems) and two were of another semantic structure (combine problems). The word problems were arrived at by manipulating variables derived from the three aspects of analysis mentioned above:

a) The Logico-mathematical aspect: Presence or absence of the quantities corresponding to the three possible sets involved. There are two possible values: 1) presence of the quantities corresponding to the two subsets; absence of the quantity corresponding to their union (this leads to an addition sentence if solved canonically). 2) Presence of the quantities corresponding to the union set and one of the subsets; absence of the quantity corresponding to the other subset (subtraction sentence if solved canonically).

b) "Real-life" situation described by the text: Two values of this variable were considered here: 1) Combine problems. 2) Change problems.

c) I. The order-of-mention (in the text) of the events: Two values of this variable were considered here: 1) The order: Time 1; (T_1); Time 2 (T_2); Time 3 (T_3), and 2) The order: T_3; T_2; T_1.

c) II. The order of mention in the text of the numerical data: Let S represent the quantity corresponding to a subset; let U represent the quantity corresponding to the union set; let ○ denote the unknown quantity. The present experiment dealt with the following values:

1) S S Ⓤ   3) S Ⓢ U
2) U S Ⓢ   4) U Ⓢ S

All possible combinations of the above-mentioned variables yield a pool of possible versions of a given story problem. Here it was decided to include those versions which allowed to contrast the relative influence of the order of occurrence of events versus their order of mention in the text.

One should note that the novelty in this study is the attempt to contrast two different 'order' variables. One variable (The order of events - OE) concerns the child's everyday experience, while the second variable (Order of numerical data in the text - OND) concerns the child's newly acquired formal knowledge, the knowledge of arithmetic and its restrictions.

Structure of Word Problems included in the study in terms of the above-mentioned variables (Order of numerical and order of events):

Problem No.	Change Problems were of the following types:		
(2)	S Ⓢ U	Order of Numerical Data	(OND)
	T_1 T_2 T_3	Order of Events	(OE)
(5)	U S Ⓢ	OND	
	T_1 T_2 T_3	OE	
(3)	U S Ⓢ	OND	
	T_3 T_2 T_1	OE	
(6)	S S Ⓤ	OND	
	T_3 T_2 T_1	OE	

Problem No.	Combine Problems were of the following types:	
(1)	S Ⓢ U	OND
(4)	U S Ⓢ	OND

The question of 'change' vs. 'combine' problems was not studied in the present research. The 'combine' problems, however, were included here in order to provide a basis for comparing those cases which represent an interaction between the order of mention of events and the position of the unknown vs. those cases in which the influence is only that of the position of the unknown.

The present hypothesis was that combine problems (in which order of events OE does not apply) would represent a task which from the cognitive point of view is in between a change problem of T_1 T_2 T_3 order and a change problem of T_3 T_2 T_1 order.

Results

The data presented herein concern two aspects of performance in verbal arithmetic problem solving: a) percentage of success for each story; b) type of algorithm chosen for each story.

Tables I and II represent the above data for grades 2 and 6.

TABLE I - SECOND GRADE

Question No.	% of Correct	Strategies (in %)		
		Rearrangement of Text	Sequential Solution	Reply One of the given numbers
1	63	2	94	4
2	64	2	91	7
3	35	20	80	0
4	81	39	61	0
5	87	7	90	3
6	35	15	80	5

TABLE II - SIXTH GRADE

Question No.	% of Correct	Strategies (in %)		
		Rearrangement of Text	Sequential Solution	Reply One of the given numbers
1	95	48	52	0
2	98	51	49	0
3	31	38	62	0
4	100	2	98	0
5	98	0	98	2
6	86	24	76	0

As seen from Tables I and II the sample of problems given represent different degrees of difficulty: the easiest one being the canonical addition change problem, in which the OND is SSⓊ and the OE is T_1 T_2 T_3. The most difficult ones being those change problems in which the time sequence is violated and the events are presented in the inverse order to that of their occurrence (T_3 T_2 T_1). Thus, the order of numerical data (OND) is seen to affect the degree of difficulty of the problems mostly for the younger Ss. This difficulty is seen to vanish for the older Ss.

Another developmental trend is readily seen when looking at the type of algorithm chosen when solving the various types of problems: the younger Ss are much more carried away by the order of the text. Whenever reorganization of the information as presented in the text is performed, it is done so that it suits the order of the events in real time. Older

Ss are less influenced by the order of the text and are more able to reorganize information in order to reach the canonical solution (influence of the logico-mathematical aspect).

The order of mention of events in the text as contrasted to their order of occurrence, appears to be the dominant factor affecting information processing in young children. The direction of the activity as described in real time (increase or decrease) and its correspondence to a canonical addition or subtraction, is the easiest case and first learned. When such a correspondence is violated and two conflicting directions are presented simultaneously to the child, a further elaboration of the text should be made to gain agreement between the conflicting pieces of information. The present study suggests that young children resolve this conflict differently to older ones.

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COMPREHENSION AND SOLUTION PROCESSES IN WORD PROBLEM SOLVING

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Many studies have been devoted to word problems but until recently they have not been explicitly related to the work on problem solving, text comprehension and learning.We present an interpretation which tries to integrate the main ideas which are thought to be important in those domains. This effort is in line with the work by RILEY and al (1981), RESNICK (1982).

A theory of vord problem solving has to integrate :

- work on procedures, especially counting procedures (CARPENTER, HIEBERT, MOSER 1981, MOSER 1982, CARPENTER 1983)
- work on the classification of problems (VERGNAUD 1976, 1982, CARPENTER, MOSER 1981, RILEY and al 1982).
- work on interpretation and representation of a problem (NEWELL, SIMON 1972, ESCARABAJAL, RICHARD 1983).
- work on simulation (RILEY et al 1982, BRIARS, LARKIN 1982)
- Work on analogy and the reference to known situations (GICK, HOLYOAK 1980
- work on text comprehension (KINTSCH, VAN DIJK 1978)
- work on learning and development of procedures (RESNICK, NECHES 1983)

1- General presentation

It is necessary to make a clear distinction between how the problem is interpreted and how the solution process is built.

Two types of knowledge play a different role in these two processes :

relational knowledge, that is knowledge about things and relations between things,procedural knowledge, that is knowledge about what to do to attain such and such goal in such and such context.

The interpretation of the situation depends exclusively of the relational knowledge. In word problems there are two types of such knowledge : general semantic knowledge, the same as for any text, and specific mathematical content, necessary to understand expression like more than, some more...

The solution process involves either procedural knowledge, specific algorithms or general strategies like counting strategies, or relational knowledge. The solution process obtained through the application of an existing procedure has to be distinguished from the solution process expressly designed for a given situation on the basis of relational knowledgo about the situation.

Procedural knowledge is expressed within procedural schemas of the type of Sacerdoti's (1977) planning nets. Relational knowledge is expressed within relational schemas of the type of semantic nets, which define relations between basic concepts.

When there is a schema fitting the situation the solution is derived form this schema and declarative knowledge associated with it. When there is no such schema, subject searches among his available procedures for one fitting the situation, as it has been coded by the interpretation process. In every case he makes the best compromise between what he understands and what he knows. In case it is impossible to fit all the aspects of the situation, as it has been encoded, then he may decide to leave some aside.

2- Basic compornents of the system

2.1 The relational structures

There are two kinds : the concepts which are the basic objects and the schemas which express relations between concepts.

Concepts :

They are concepts which imply only general semantic knowledge : the concept of set (of course not in the mathematical sense) and the concept of transfer between sets. The other are mathematical concepts : they are defined within schemas.

The concept is a set of free slots which have to be instantiated so that a meaning may be assigned to an object through the concept, except for some which are optional : if there is no information fitting the slot, the slot is meant as irrelevent in the present case.

For instance the concept of set is defined by

- au object type
- au identifier : the owner of the objects, the place where they are...
- a number (the number of elements)
- (optional) a time index(t_0, t_1) indicating the moment at which the other specifications of the set are true. This information is a qualifier of the identifier and of course does not exist for static problems

The concept of transfer is defined by

- a destination set, an origin set or both
- an object type
- a time reference relative to t_0, t_1 (before, after, past tense...).
- a number

Schemas

A schema expresses an operation on two sets and its result, which is a set, with a relation as an identifier.

For instance the transformation schema is defined as follows :

	first set (initial set)	second set (final set)	result set (change)
object type	a	a	a
identifier	X to	X t1	has more (X t1, to)
number	n1	n2	n2-n1

We consider the following schemas presented in the order of acquisition. We give only for each set the form of the identifier.

	first set	second set	result set
part -whole	X	Y	X and X together
more or less comparaison	X	Y	Who has more(X,Y) less
whole-part	X et Y together (or superordinate of X and Y)	X	Y
transformation	has (X to)	has (X t1)	how many more(Xt1, less X to)
how many more comparaison	X	Y	how many more (X,Y) less
composition of transformations	hes more (X t1,Xto) less	has more (X t2,Xt1) less	how many more (Xt2,Xto) less

We distinguish between part-whole end whole-part schemas because it is easier to infer the supraordinate from the subordinates (to infer that boys and girls are children) than to infer the subordinates from the supraordinate (to infer that children are either boys or girls).

2.2 The procedural structures : procedures

The procedures we consider are those described by CARPENTER and MOSER (CARPENTER and al 1981, MOSER 1982, CARPENTER 1983)- Three levels are distinguished : procedures applied to objects and modeling action, counting procedures, number facts. We describe some of threm as exemples :

Direct modeling procedures	counting procedures
Joining : adds n elements	counting on : begins with the

to an existing set of m elements and counts the number of elements in the two sets	number m (or n) and adds one n (or m) times
adding on : adds elements to an existing set until a given total b is attained and counts the number of added elements (n)	counting up from given : begins with m and goes on forward in the counting sequence until b as attained, while initiating a second count from 1. The answer is the number attained in the second count
separating from : removes n elements from an existing set of m elements and counts the remaining elements	counting down from : counts backward beginning from m n times. The answer is the number attained in the backward sequence.

We retain the idea proposed by CARPENTER (1983) that there is a correspondence between counting strategies and strategies applied to objects. Moreover we suggest that there is also a parallelism between number facts and counting sequence, retrieving from memory a number fact being a shortcut for a counting sequence.

We describe a procedure by a procedural schema which is very close to the description to SACERDOTI'S planning nets (1977) and to the description adopted by RILEY and al (1981)

A procedure is defined by

- a sequence of actions (body of the procedure)
- a goal : the result of the sequence
- a set of preconditions : the prerequisits which must be present so that the procedure many be run.

An available procedure is activated when among the elements produced by the interpretation process are present both its preconditions and a question corresponding to its goal.

3. The solving process

3.1- interpretation process

At the first level primitive object frames available to the subject are instantiated from semantic analysis of the text. Then subject tries to integrate these pieces of information into a schema by matching first-level interpretation with available schemas. When there is no available schema, there is no second-level interpretation and the only thing subject can do is to try to activate a procedure, or make inferences allowing a procedure to be activated (for instance in a joining/result unknown problem, knowing that there has been a transfer of n objects to x after t_0, he may infer that x has a set of n objects which is different from the first set, so that the joining or counting on procedure may be applied.

At this level misinterpretations may occur if the adequate schema is not available. For instance for a subject having the transformation schema but not the composition scheme a problem such as "Jim gained 8 marbles then lost 5. How many marbles has he gained at the end ?" will be interpreted as "Jim had 8 marbles, he lost 5. How many the has at the end ?". That such misinterpretations are plausible is confirmed by the fact that when subject is requested to formulate another problem with similar by changing only the contexts, he builds a change start unknown problem. (ESCARABAJAL in preparation) Another exemple of misinterpretation is to interpret the question of a compare problem "how many more marbles does John have than Peter" as "who has more ? How many does he has ?". This will be obtained if the subject has not the how many more comparaison schema but only the who has more comparaison schema : this is the best interpretation possible in this case.

3.2- solution process

When second level interpretation succeeds a solution process is built through knowledge of relations associated with schemas. When there is no appropriate relation, subject looks for knouwn procedures fitting the schema.

When the second level interpretation fails, only the information available at the first level many be used. Then subject search for a procedure fitting the information about the problem. If one is activated, a new set is produced and is added to the information about the problem.

When all possible procedures are applied subject tries to match each set present with the set described in the question. The set that best mathes is chosen In this case the solution process is data driven; when it is derived form a schema, it is structure driven.

4- The learning process.

Learning is supposed to take place at 3 levels :

-modifications of the sequence of actions withen a procedure

-learning of new procedures

-learning of new schemas

We shall develop this point in the oral presentation.

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YOUNG CHILDREN'S COMPREHENSION OF "MORE" AND "LESS" IN SIMPLE COMPARISON PROBLEMS.

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Research on semantic development has provided evidence of the various difficulties that children undergo before acquiring full comprehension of relational terminology (Clark H.H., 1970 ; Mc Neill, 1970 ; Clark E.V., 1972). Furthermore relational terms have been at the center of studies concerned with children's development of dimensional and mathematical concepts when these are evaluated using verbal methods (Mehler and Bever, 1967 ; Beilin, 1968 ; Siegel, 1977 ; Jones, 1982).

An important example of these terms are the words "more" and "less". The present work was designed to study young children's comprehension of "more" and "less" when they are to solve simple comparison problems. Earlier studies on pre-school children (Donaldson and Balfour, 1968 ; Palermo, 1973, 1974 ; Weiner, 1974) have indicated that understanding of "more" and "less" is acquired gradually. Initially both terms are interpreted by children as meaning "amount or quantity of". Their comparative sens will be acquired later with "more" being understood before "less". Finally comparative and contrastive meanings are grasped.

Most of these semantic studies have tested comprehension of the terms presenting arrays of objects in different number to be compared though little attention is given to young children's capacities for establishing correct quantity comparisons.

Number of studies have found regularities in children's understanding of quantity dimensions, mainly an increasing ability to differentiate and coordinate length, density and number (Gelman, 1972 ; Lawson, Baron and Siegel, 1974 ; Shannon, 1978). Young children's capacity to establish correct comparison of quantity is also acquired gradually. Initially children use length, extent or visual one to one correspondence when judging small numbers. After they are able to solve problems where these variables give no cue to number and finally solve problems in which number and length are negatively correlated.

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Therefore if children have problems with "more" and "less" and at the same time their quantity comparison abilities, depend on dimensions they do not master, problems on performance could relie on one or both aspects. Our purpose was to determine if the comprehension of "more" and "less" in comparison tasks is related to the complexity of the cognitive operations required of children.

EXPERIMENT I (DISCRIMINATION TASKS "SMALL QUANTITIES")

Subjects were 45 pre-school children attending 3 groups of public french school. 15 children per age of 3, 4 and 5 years composed each group. Mean ages were 3.7, 4.7 and 5.6.

Material consisted of 5 cards (10 x 15 cms). Each card had two linear arrays of dots, each line of different color arranged in a one to one correspondance. Number of dots on the sets resulted from all possible combinations of numbers 1 through 6 providing numerical differences of 1 to 5 elements.

Interviews were individual. Cards were presented one by one to Ss. During a first session only "more" questions were proposed. A second session one month later presented "less" questions.The exp. gave instructions in the following terms : -Look at the card carefully and "show me the line with "more" ("less") dots.

RESULTS

An analysis of the percentage of Ss that failed on each numerical difference between comparison sets, is shown on table 1. It can be observed that the number of Ss with error decreases with age for both terms. An important difference appears on the number of Ss with error comparing performance for "more" to that of "less" in all age groups. Notice that no significant relation can be establish among the number of Ss with error and a certain numerical difference between sets.

TABLE 1 : Percentage of Ss with error for each numerical difference between sets of comparison on "more" and "less" discrimination tasks (small quantities)

Elements of difference between sets		1	2	3	4	5
3 years (n=15)	"more"	20 %	13 %	33 %	33 %	26 %
	"less"	80 %	66 %	66 %	80 %	80 %
4 years (n=15)	"more"	20 %	0 %	6 %	6 %	6%
	"less"	60 %	53 %	60 %	66 %	66 %
5 years (n=15)	"more"	0 %	0 %	0 %	0 %	0 %
	"less"	13 %	13 %	13 %	13 %	13 %

EXPERIMENT II (DISCRIMINATION TASKS "BIG QUANTITIES")

Subjects were 40 pre-school children attending two groups of public french school. 20 children per age 3 and 4 composed each group. Mean ages were 3.4 and 4.5. Material consisted of 6 cards (10 x 15 cms). Each card had two sets of dots, each group of different color gathered in a bunch. Number of dots on the sets was established to provided numerical differences of 10,11, 12, 13, 14 and 15 elements. The small set never depassed 5 elements, Interviews were individual. Ss were given "more" or "less" questions at random, and 4 groups were established. One group of each age level answered "more" questions and this was the same for "less". Procedure was identical to that used on Experiment I.

RESULTS

An analysis of the percentage of Ss that failed on each numerical difference per age group is shown on table 2. It can be observed that (like in Experiment I) an important difference appears on the percentage of Ss with error when comparing performance for "more" questions to that for "less" questions in both groups. Specially for "less" age level increases correct responses.

Again no significant relation can be establish among Ss with error and a certain numerical difference.

TABLE 2 : Percentage of Ss with error for each numerical difference between sets of comparison on "more" and "less" discrimination tasks (big quantities).

Elements of differences between sets		10	11	12	13	14	15
3 years	"more" (n=10)	10 %	10 %	20 %	10 %	10 %	10 %
	"less" (n=10)	60 %	80 %	80 %	90 %	80 %	70 %
4 years	"more" (n=10)	10 %	10 %	10 %	10 %	0 %	10 %
	"less" (n=10)	60 %	60 %	40 %	50 %	50 %	40 %

DISCUSSION

From our results it appears that children's understanding of "more" and "less" in simple comparison tasks increases with age and is acquired gradually. This is consistent with earlier findings assessing these in preschool childrens (Donaldson and Balfour, 1968 ; Palermo, 1973, 1974).

The difference on children's performance when comparing "more" tasks with those for "less", shows that "more" is understood before "less". This confirms the asymetry phenomenon of acquisition of certain relational terms observed in other works (Clark, H., 1970 ; Clark, E.V., 1973). Furthermore children's errors in our study provided evidence of a confusion of the term "less" with the term "more". This was first reported by Donaldson and Balfour (1968) and made them suggest a possible period in the acquisition were "more" and "less" would have a synonimous sens for children.

It has been suggested elsewhere (Siegel, 1977) that discrimination of dimensions related to quantity can be assumed to provided the basis for the assigment of meaning for words like "big" and "little". Our results indicate that children's comprehension of "more" and "less" cannot be attributed to numerical differences in the arrays to be compared. Still more even providing children with developmentally important cues (quantity positively correlated with length or extent, one to one correspondance) concerning comparison abilities which we thought would alow early notions of the terms to emerge, children's performance does not clarify Siegel's assumption.

Finally, though our results confirm a number of earlier observations, they still raise the question of which can be the criterial features that children consider in order to grasp particular meaning of "more" and "less" in quantity comparison tasks. Further research in this type of knowledge seems to us of great importance on child's psychological research using verbal methods and mathematical instruction.

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PROBLEM SOLVING AT ELEMENTARY SCHOOL

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In the year 1981, an experiment was conducted to evaluate children's ability to related different formulations of a simple arithmetic problem.

The following pairs of problems were presented to 573 students, with at least 4 years of scholarity, aged 9 to 17, of schools nearby the University Campus:

A_1

If a □ is worth 3 ○ and
a ○ is worth 2 △ then
Conclude:
a □ is worth .... △

A_2

John is three times older than Mary, and Mary is twice older than Peter. Peter is two years old. Conclude:
John is .... years old.

Results

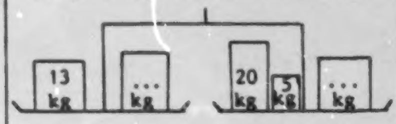
A_1 ↓ A_2 →	Correct	Wrong	Non response	Total
Correct	12 (2,1%)	8 (1,4%)	0 (0,0%)	20 (3,5%)
Wrong	279 (48,7%)	150 (26,2%)	37 (6,4%)	466 (81,3%)
Non response	33 (5,8%)	27 (4,7%)	27 (4,7%)	87 (15,2%)
Total	324 (56,5%)	185 (32,8%)	64 (11,2%)	573 (100,0%)

These results indicate that children were unable:

- to understand the symbolic form of the problem A_1;
- to relate problems A_1 and A_2.

B_1

The pair of scales is balanced.



Write the missing number on each plate.

B_2

Write the missing numbers to obtain an equality:

$$7 + 8 + \ldots = 10 + \ldots .$$

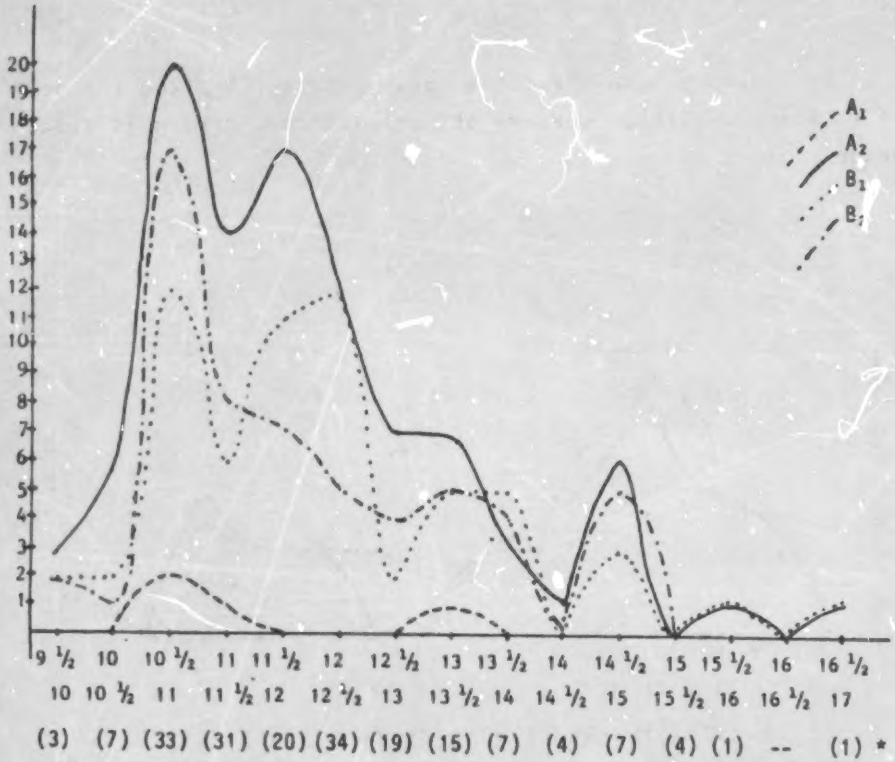
Results

B_1 ↓ B_2 →	Correct	Wrong	Non response	Total
Correct	109 (19,0%)	93 (16,2%)	19 (3,3%)	221 (38,6%)
Wrong	58 (10,1%)	166 (29,0%)	31 (5,4%)	255 (44,5%)
Non Response	19 (3,3%)	54 (9,4%)	24 (4,2%)	97 (16,9%)
Total	189 (32,5%)	313 (54,6%)	74 (12,9%)	573 (100,0%)

Many children were unable to choose one out of the possible solutions.

Nine mathematics teachers at IM/UFRJ have worked at this experiment.

A randon sample of 186 individuals was selected from the entire universe of 573 students. The idea was to relate the performance of these students in problem solving with their age. We obtained the following distribution.



* Total of students.

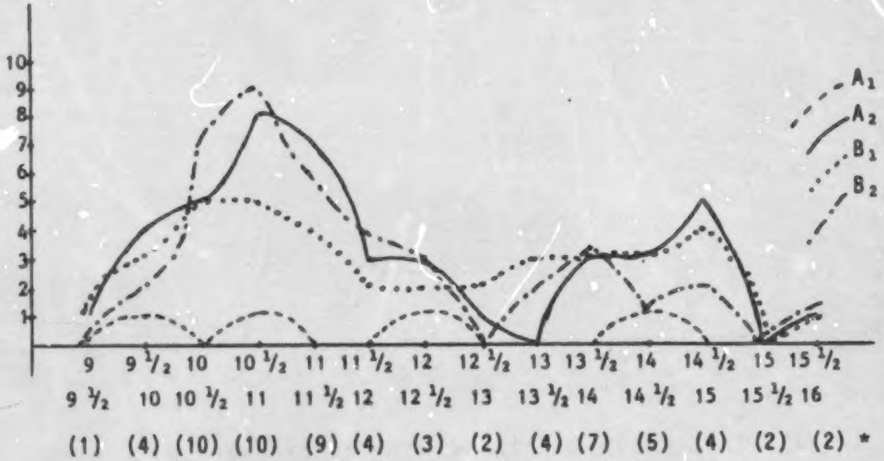
Conclusions:

. The best performance in solving problem A_1 was obtained by the children aged 10 ½ to 11 ½ .

. Problem A_2 was solved by the majority of children aged 10 ½ to 12.

. The majority younger children wich regular performance at school solved problem B_2.

In order to understand the above results, I asked the help of a secondary school teacher of mathematics working at a 5th grade class.



* Total of students.

In april 1983, the secondary teacher repeated the experiment at two other 5th grade classes, where questions A_1 and A_2 where presented in the reverse order. This experiments has shown that no sensible modifications of the performance of the students were detected, as the above distribution suggests.

"WHO 'S GOT THE HIGHEST NUMBER?"
THE CONSTRUCTION OF A DIDACTIC SITUATION IN FIRST YEAR ELEMENTARY CLASS

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This study is part of research being carried out, in collaboration with Annie Bessot, into mathematical didactics and whose aim is to clarify the conditions under which the learner constructs and interiorizes the concept of natural number, and to show the learner's different conceptions functioning at a given time and within a given situation.

Having already spent a long time on the design, implementation and analysis of one-to-one interviews (cf. bibliography) it was decided to adopt a different approach; namely to intervene in the class itself and to construct didactic situations in order to explore , via specified learning tasks, conceptual fields which were of interest to us.

The learning task for which we were responsible, and which is presented below, was to establish and make operational the link existing between writing numbers within the decimal system and the number order. The situation which the present paper describes is 2nd year elementary class (7-8 year-olds) at time of the school year when the teacher had already carried out revision of comparison of numbers via the written form. The learners, therefore, had already had to solve problems of comparison of two or more numbers.

The situation in which we put them consists in the comparison of two numbers, one of which is known and the other unknown.

I. DESCRIPTION OF THE PROBLEM SITUATION.

- The teacher chooses two numbers (N1) and (N2). She writes each number on seperate cards.

- (A) plays ((against)) (B). A is given a card on which (N1) is written. (B) is given the card (N2). A and B are mutually unaware of the number on the other's card.

- The game consists in finding out, in the shortest possible time, whether A or B has the highest number.

(1) A asks B a (written) question. B poses a (written) question to A. The questions are exchanged only when A and B are both satisfied with their question.

(2) A gives a (written) answer to B and this is reciprocated. The exchange of answers is simultaneous.

(3) After examining the answers that they have received A and B ask, when necessary, a further question... and so on until one claims to know "who has the higtest number".

(4) He then explains to his partner in writing (and in the presence of the teacher) the grounds for his claim.

(5) A and B may then reciprocally show their numbers on their cards and check whether the claim is well founded.

(6) The game is played (N) times.

Remark.

This situation differs from conventional situations of comparison in several ways.

The learner is obliged to search for information about the unknown number. In order to do this he is obliged to make hypotheses about it in ordor to be able to ask pertinent questions in the light of his present conception of number. This obliges him to formulate a series of questions which enable him to compare the unknown number to his own. This series itself becomes an object of reflection as the game goes on, as the aim is to find the highest number as quickly as possible.

The learner is in a position to organize his own time as the exchange of questions between A and B occurs only when both are satisfied with the question that they have written. Each game comes to an end when either A or B claims to have the highest number.

II. VARIABLES OF THE PROBLEME SITUATION. (Didactic variables)

2.1 In the description that has been made the words between brackets can be considered as variables. certain of these have been fixed (double brackets -(( ))-) for the following reasons :

a) A is made to play against B (and not with) in order to eliminate questions which would reveal too much information to the partner concerning the unknown number. The learner is thus forced to think the question he will ask.

b) A situation of written communication (not oral) between A and B has been chosen. Thus,

- The questions and answers are recuperable.
- There is interaction between the strategies of A and B as each must read the other's question in order to answer it.
- each learner can recap, thus facilitating insights, restructuring and the elaboration of new questions.

The vaiables (single brackets - ( ) -) have been given different values in order to obtain different strategies, ways of adapting and controlling. That is to say, learning. These are the didactic variables of the situation and they enable us to set up a series of situations to lead to the defined learning goal.

2.2. A priori study of different possible strategies and their relationships to the different values attributed to the variables (N1) and (N2).

There are two major strategies.

a. Those where the learner attempts to find the unknown number in order to compare it with his own. In other words, the learner tries to return to the situation of the comparison of two given numbers. All strategies of this type will be coded T (first letter of the word "trouver" in French).

b. The other strategy is where the learner asks questions whose answers will enable certain features of the unknown number to be ascertained. The comparison of the two numbers can be deduced from this without the hidden number being known.

A certain number of strategies link up with these two main types. Their efficiency and reliability vary, both according to the values given to the variables N1 and N2, but also in function of the knowledge system of the learner. In particular, it seems reasonable to suppose that the change in the numerical field may, by increasing the difficulties of type T strategies, lead the learner to resort to different properties of numbers, if he is in a position to do so, and to the elaboration of more efficient and more reliable strategies.

2.3. Role of interaction variables.

We have, it will be remembered, defined the type of interaction between two competitors. However, one may choose either, to make one learner compete against another etc... In this way, one can either constrain or encourage another type of interaction, namely, learner cooperation within a group in order to attain general agreement on the choice of a satisfactory question.

Thus, the selection for A and B of a group of learners cooperating will diversify the interaction and this will encourage the generation of different questions, their confrontation and thus the elaboration of a group strategy.

If, on the other hand, the game is between two learners, the interaction of A against B will not in itself encourage the elaboration and development of strategies unless it creates a conflict between two differing points of view. This means therefore, that a choice of pairs must be based on prior analysis of the knowledge structures of each competitor.

It should be added that the advantage of this approach is that it leaves each learner master of his own strategy.

2.4. Role of N.

As to the variable N, that is to say, the number of games played, clearly, the greater the number is, the greater the opportunities afforded to the learner to envisage, to encounter and thus to develop new strategies are. The teacher can determine this number herself, but she can also leave it to A and B to decide whether to replay or not.

DURING THE ORAL PRESENTATION OF THIS PAPER I WILL SHOW THE DIFFERENT APPROACHES TO BE FOUND IN A SECOND YEAR ELEMENTARY CLASS AND I WILL ATTEMPT TO SHOW THE EFFECT OF THE DIFFERENT TYPES OF INTERACTIONS AND WHAT WAS LEARNT.

SPECIAL DIFFICULTIES OF ARAB PUPILS WITH NUMBERS RESULTING FROM THE RULES OF THE ARABIC

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§1. INTRODUCTION

Counting methods and the concept of number are the basis of elementary school arithmetic. They form an important part of the curriculum and much time and effort are devoted to them. Nevertheless, young children have difficulties in writing and reading numbers. This is because of the complexity of the decimal structure on one hand and the children's inability of systematic thought on the other hand. Errors in writing and reading numbers are widely spread all over the world. Ginsburg, 1977, claims that children "write numbers as they hear them" and thus they write "402" when hearing "forty two" (p. 89). This is claimed about children in the western culture. But Arab children face more difficulties than western children because of the special structure of the Arab language. Not only that it is written from the right to the left (this is also true about the Hebrew) but there are also systematic disagreements between the number name in Arabic and the decimal structure. For instance, 345 is read in Arabic: three hundreds and five and forty; 68273 is read: eight and sixty-thousand and two hundreds and three and seventy. As a result of that Arab pupils in the fourth grade have difficulties in writing numbers in the decimal form. Also, they have difficulties in writing numbers in the decimal form when given to them in Arabic.

§ 2. IDENTIFYING NUMBERS

The following questions were given to 344 pupils in grade 4. (We write the number names in the order the words appear in Arabic.)

1. Which of the following numbers is "four hundred, one and thirty"

   A. 31400 B. 314 C. 431 D. 40031

2. Which of the following numbers is "seven hundred and nine"

   A. 7009 B. 9700 C. 907 D. 709

The results are given in the following two tables.

Table 1 - Distribution of the answers to question 1 (N = 344)

A. 31400	B. 314	C. 431	D. 40031	No answer
10%	14%	55%	15%	6%

Table 2 - Distribution of answers to question 2 (N = 344)

B. 9700	C. 907	D. 709	A. 7009	No answer
11%	19%	50%	14%	6%

We changed the order of the distractors in question 2 so that they will fit the order of the distractors in question 1. When analysing the errors one can see that each of them has its own logic. In 1A and 2B the pupil identifies the hundreds correctly (400 or 700) and then goes from the right to the left and identifies the rest of the number as expressed in Arabic (one and thirty or just nine). Thus a lack of understanding of the decimal structure together with the influence of the order of words in Arabic are active in the formation of this error. A lack of understanding of the decimal structure in this case means that the pupil identifies (or writes) the number exactly as he hears it and this is the mistake in Ginsburg 1977 mentioned above.

In 1B and 2C the pupil is probably aware of the decimal structure but he identifies the digits from the right to the left in the order they appear in the Arabic number name (four hundreds, one and thirty).

In 1D and 2A the pupil has overcome the order problem but he is still lacking the understanding of the decimal structure.

Note that in both questions the pupil was asked to identify a number and not to write down a number. This can explain some "inconsistencies' in the distractors. For instance, if the pupil writes from the right to the left "four hundred, one and thirty" he might write 301400. However, since questions 1 and 2 were identification tasks it might be wrong to conclude that all the pupils will necessarily write the numbers 431 or 709 in one of the four ways given in each question.

The difficulty in identifying numbers increases when more digits are involved. The following question was given both to 4 and 6 graders.

3. Which of the following numbers is "seven and fifty thousands and five hundred and seven.

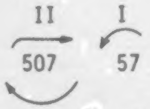
   A. 57507 B. 57000507 C 570757 D. 575007

   The results are in table 3.

Table 3 - Distribution of answers to question 3

	A. 57507	B. 57000507	C. 50757	D. 575007	No answer
4th graders (N = 344)	40%	23%	10%	16%	14%
6th graders (N = 286)	62%	16%	4%	12%	4%

Here a new type of error is introduced in C. The pupil changes the right order twice. He identifies "seven and fifty thousand" as 57 at the right side (probably he reads it from the right to the left). Then he goes to the left side and reads from left to right "five hundred and seven" as 507. This is demonstrated by the following arrow diagram:



So, contrary to the western culture, the Arab children are not only confused with the decimal structure, they are confused also with the writing and reading directions and sometimes they change the reading direction twice in the same number.

§3. ARITHMETICAL OPERATIONS

The above confusion has an enormous effect on the ability to carry on correctly arithmetical operations. Two examples for that. The 344 fourth graders in the above sample got the following exercise: Add the following numbers: 3, 42, 100, 51. Only 54% added the numbers correctly. The most common errors were:

(a) $$\begin{array}{r}51\\100\\42\\3\\\hline 1330\end{array}$$

(b) $$\begin{array}{r}3\\42\\100\\51\\\hline 223\end{array}$$

(c) $$\begin{array}{r}51\\42\\100\\3\\\hline 1060\end{array}$$

Although western children also make similar mistakes the percentage here is much higher. This can be explained by the following:
The analysis of 3C in § 2 suggested that some Arab children lack a "reference point" when reading or writing a number (since they can start from the right to the left and then go all the way through and start from the left to the right). This fact might cause them inconsistencies when writing numbers in columns. The column (a) above is incorrect but at least it is systematic. Columns (b) and (c) are both incorrect and not systematic and perhaps the above lack of reference point is a cause for that. Although also Ginsburg, 1977, reported about non-systematic performance of the above type (p.113) it seems that it occurs more often in Arab children.

Another example which demonstrates the confusion mentioned in § 2 is the following: 255 fourth graders were asked to divide 256 by 16. Only 38% did it right. About 20% did the following mistake: They used the long division algorithm and divided 25 by 16. They wrote 1, multiplied, subtracted and transferred the 6 as required; then they divided 96 by 16 and wrote down the result (6) at the left side of the 1. Thus they got:

$$\begin{array}{r} 61 \\ 16\overline{)256} \\ \underline{16} \\ 96 \\ \underline{96} \\ -- \end{array}$$

§4. THE DECIMAL STRUCTURE

The two following two questions are related directly to the decimal structure.

4. What will happen to the number 237 if we write down in front of it 4 (and thus obtain 4237)?
   - A. The number will increase by 4.
   - B. The number will be multiplied by 4.
   - C. The number will be increased by 4,000.
   - D. The number will be multiplied by 10.

5. What is the value of the digit 2 in the number 182506?
   A. 2 B. 2000 C. 200 D. 20000

The results are given in tables 4 and 5.

Table 4 - Distribution of answers to question 4

	A	B	C	D	No answer
4th graders (N = 344)	17%	9%	48%	12%	14%
6th graders (N = 286)	8%	5%	75%	7%	5%

Table 5 - Distribution of answers to question 5

	A	B	C	D	No answer
4th graders (N = 344)	13%	55%	12%	9%	11%
6th graders (N = 286)	6%	73%	6%	13%	2%

§5. SUMMARY

We have discussed some difficulties of Arab children in arithmetic. These difficulties are specific to the children as Arabic speakers. The trouble is that many teachers are unaware of these specific difficulties and this fact does not help to solve the problem. Although we do not yet have a solution to the above difficulties we believe that teacher awareness to the problem will be a first step in the right direction.

References:

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NUMBERS IN CONTEXTUAL FRAMEWORKS*

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Summary

This contribution consists of a report on research into the use of the bus-context (boarding and leaving the bus) when introducing addition and subtraction in the first-grade.

Other contexts are taken into account as well, such as skittles, magic tricks, birthday candles, rides in a "cart", distribution of passengers on a double-decker bus, number-cards and ages.

These contexts influence the use of numbers and operations. This implies that numbers used for instance within the bus-context will have other properties from numbers used to describe skittles.

Apart from the sound, bus-numbers are not the same thing as skittles-numbers. Therefore. it makes sense to speak of numbers in a contextual framework. It is suggested that these contexts first be offered to the children separately before making comparisons between them.

Finally, an outline is made of misconceptions which can arise regarding the bus-numbers or due to a luck of context.

I. INTRODUCTION

a. Situations and contexts

Numbers and operations are connected to various arithmetical situations and contexts. These situations and contexts determine the limitations in the use of the numbers.

Examples:

- A six year old girl blurted out: "I'm six, but in the train I'm three."
- Ikos, an 11 year old boy came in furiously, saying: "That's not fair; together they're 22." He meant the two 11 year old boys who were fighting with another boy who was "only" twelve.

Evidently, we must keep the numbers closely connected to the context and situation in which we use them in order to determine what one can do with the numbers. In any case, it does not go without saying that one can add, for example, ages together, just because one can add up numbers.

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* A more extended paper will be available at the conference.

b. I.O.W.O

Along side the use of numbers in specific arithmetical situations (number-cards, number-line, counting-frame, abacus, basic sums such as 2+2=4, etc.), various contexts have been devised for mathematical activities. The now defunct I.O.W.O. (Institute for Development of Mathematics Education) had developed a number of such contexts (see Freudenthal (1976); Van den Brink (1980)).

The contexts have in common the fact that they are not simply illustrative of the operations to be learned. A given context has its own structure which sets limitations on the use of numbers which are specific for that context.

c. Education

In traditional mathematics education however, this viewpoint is of secondary importance. Contexts are purely illustrative; one after the other, dots, squares, dashes, strawberries, etc. are used indiscriminately, as if they were interchangeable. This is done in order to give the children the opportunity to see the numbers and their operations as abstractions. The children, however, often find it a joke:

The mother of 6 year old Kikkie recalls, "There were 6 of us at home: uncle Wim, Gerrit, Eef and Piet, aunt Alie and myself." "6? There were 6? That's funny", says Kikkie, with a 'where've-I-heard-that-before' face. "oh, I know: I'm 6, hahahaha! Funny, heh?"

The contexts we choose should not be purely illustrative. They need to have been already experienced by children; things like magic tricks, ages, games of marbles or skittles, etc. In brief, all sorts of coöperative and play situations. First-grade arithmetic should consist of play activities: play-acting, games, etc. (Van den Brink, 1980). These contexts can be used later for reference in the clarification of algorithms.

The context - that is, the meaningful association in which the number is used - determines whether the substitution of one number for another can be allowed. On the other hand, the substitution of contexts around a specific number or operation is a possible manner of leading up to abstractions.

Both abstraction and substitution have their drawbacks: abstraction can lead children to blindly adding up apples and oranges; substitution can give children the wrong idea mathematically by using a too limited context. We must therefore set requirements for the contexts and make sure of appropriate contexts for introducing and applying mathematical situations. I.e. we must examine the contexts to see what limitations they may impose upon context-*free* numbers and operations.

d. Numbers

The consequences of this point of view are found in numbers and operations and even in the description of them.

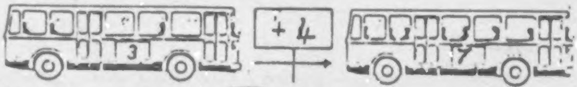
In literature, the use of numbers is described with terminology borrowed from mathematics: cardinal and ordinal numbers, measuring numbers, numerals, etc. This assumes implicitly that research is being done exclusively in this area of number use, even though everyone knows that children and mathematicians deal with numbers in a much more complex fashion.

In order to gain psychological knowledge and insight into a child's view of the concept of numbers, I prefer to speak of *age-nubbers*, *bus-numbers*, *skittles-numbers*, *marbles-numbers*, *number-cards*, *magic-numbers*, the number "2+2=4", all of which are connected with specific contexts or situations.

II. BUS-CONTEXT and BUS-NUMBERS

a. Bus-context

One of the contexts which was developed by the I.O.W.O. and which is now being researched further by the OW & OC department of the University of Utrecht is that of the bus. The bus can be put to good use in introducing addition and subtraction. A simple illustration is enough to give an idea:



This *simple bus-sum* consists of only one bus-stop, as opposed to *"chain-sums"* which show a complete bus-ride. In similar problems three numbers are mentioned which we shall call:

- the *starter* - the number of passengers on the bus before the bus-stop (in this example: 3).
- the *operator* - the + or - sign followed by a number on the signpost at the bus-stop (+4) and an arrow pointing in the direction of the bus.
- the *resultant* - the number of passengers left on the bus after the bus-stop.

The first mention of the bus-model was in 1974 (Van den Brink). There were various reasons for the introduction of this model:

- Firstly, the designer required a model for introducing addition and subtraction. The bus-model satisfies this requirement.
- Children's informal sums call more for resultant operating (as in the bus-model) than for comparitive operating (Ginsburg (1977); Hercovics et Bergeron (1982)).
- The use of arrow-notation refers to actual bus situations; that is, the situation is always recognizeable in the notation (Vergnaud (1982),(Kolb and Clay (1982)).
- Arrow-notation gives the opportunity to record information about events in other contexts than that of the bus, for instance information about a game of skittles or marbles outside of school. Not only is the social function of arrow-notation contained here, but also the possibility of substituting contexts.
- A large variety of practice forms are possible with arrow-notation.
- Etc.

b. Bus-numbers

With regard to context-free numbers, bus-numbers have a few specific characteristics:

a. no negative numbers
b. no unlimitedly large numbers
c. simultaneous addition and subtraction
d. repetitive addition and/or subtraction
e. non-commutative for addition
f. directional problems in reading and bus-travel
g. relation between starter, operator and resultant
h. the driver
i. the fixed route
j. other contexts and number-systems, but only one arrow-notation

An extended explanation will be given at the conference.

III. HYPOTHESES FORMED BY RESEARCH INTO THE BUS WITH KINDERGARTENERS

We first wished to find out whether young children - 4 to 6 - were familiar with the bus-context as we see it. One of the bus's charadteristics is that it makes continuous stops whereby the number of passengers it is carrying varies. This is similar to a train, for instance; or to the results in a game of skittles which also vary continually. Children however, are not always aware of this changing structure regarding the bus. They are more apt to think up a variety of other emotionally tinged stories and ideas about the bus, the train, skittles, or the "cart" in which they ride around the playground.

We shall list some of these ideas, and emphasize the elements which can be important in mathematics instruction. Under discussion are: real buses (destinations and intermediate stops, buying tickets and transfering) and the bus as a toy and as a game.

The following remarks compare the models which were discussed

a. The bus makes *intermediate* stops.
This is unlike the *schoolbus* which only stops at the destination and at traffic-lights. It is also unlike *bowling*. On the other hand, the *toy train* is similarly repetitive, as is *skittles*.

b. There is only *one driver* on the bus; this is unlike the cart, where the drivers take turns.

c. The *bus's fixed route* is shared by the *toy train* but not by the *cart*.

d. At every bus-stop the bus can be *boarded as well as left*. This is not the case with *skittles*; there the pins are only knocked down with each throw, corresponding to only one of the main operations: either addition (of points) or subtraction (the number of fallen pins).

It is clear that the above models differ considerably from each other; so much that, in the eyes of kindergarteners, they may even have nothing in common.

But that is true as well for the numbers involved. They differ too, as what is allowed with numbers in one context is not possible with numbers in another context. This is for that matter characteristic of "doing maths".

But it should also be characteristic of kindergarten and first-grade instruction to let children play with numbers in various *separated* contexts in the form of assorted *games*, without always prefering to work *abstractly* by insistently pointing to the similarities in the *names* of the numbers.

At this age, arithmetic should be a *playing* of social games, the results of which can be recorded in *arrow-notation*. In doing so, we should however be aware that one context approaches the desired object of the instruction better than another.

IV HYPOTHESES FORMED BY RESEARCH INTO THE BUS IN FIRST-GRADE

Graphic bus in first-grade

Introduction

Along with children's stories and ideas about the bus, important rôles in this subject in first-grade are played by series of drawings of buses and by arrow-notation.

Major mistakes or misconceptions regarding the graphic bus are the following:

1. Extremes in occupancy: a completely filled or an empty bus as starting-point.
2. The result used as the operator
3. Boarding and leaving at one and the same bus-stop.
4. The exchange mistake: the bus is seen as bus-stop sign and vice-versa.
5. Mistake by doing the sum

We reviewed the kind of instruction the children were receiving at the moment that we noticed these mistakes. By this means could we determine the following:

- during the bus-games in January (skittles, marbles, play-acting the bus, etc), the children had the impossible problems in hand.
- However, as soon as notation was brought up during the bus-stories in February, the following mistakes appeared:
  - the resultant used as an operator
  - buses exchanged for bus-stops
  - mistakes in doing the sum
- It is striking that these three types of mistakes disappeared during the phase where various arrow-sums were introduced for practice in March.
- But the exchange mistake reappeared some months later - in June - when the children were busy doing sums without utilizing the contexts. This led to misconceptions which could again be resolved by refering to the contexts.

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See the paper available at the conference.

CHILDREN'S PERCEPTION OF FRACTIONS AND RATIOS IN GRADE 5

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Thomas R. Post
University of Minnesota

THE PROBLEM OF BEGINNING FRACTION INSTRUCTION WITH EMBODIMENTS

Common methods of fraction instruction use models that embody the idea of fraction; for example, regional models support fractions as parts of an equipartitioned whole, or counting models view fractions as a subset of a given whole set of counting items. Advantages of such approaches are that fractional ideas are based on concrete, imaginal representations which are related to the child's reality (e.g., pizza models). The disadvantage is that the conception of fraction derived from such models relies on a specified whole unit. Rather than thinking of 2/3 as a number, that is, of a quantity like 1 or 2, children would likely think of 2/3 of a whole unit. Likewise, in the context of fractions children tend to speak of whole numbers as 1 whole, 2 wholes, etc. The question emerges whether an instruction what uses part-whole embodiments really can provide a foundation for a concept of (positive) rational number. Current curricula very early call for application of rational number concepts, for example, in proportional reasoning. Proportional situations require an understanding of fraction which is independent of fixed units. An instruction aimed toward developing an understanding of rational number thus needs to deal with the question of how to achieve students' independence of fixed units for fractions.

THE EXPERIMENTAL INSTRUCTION IN THE RATIONAL NUMBER PROJECT

The Rational Number Project is a multi-site effort funded by the National Science Foundation from 1979 through 1983. Instruction in a 30-week teaching experiment in 1982-83 was based on the multiple-embodiment principle (Dienes, 1971) and included the use of several types of manipulative materials, representational modes, and rational-number constructs (Behr et al., 1980). An important aspect was that translations between different modes of representation which subjects frequently were to make during instruction would facilitate the abstraction of rational-number ideas (Lesh et al., 1980). One focus of the project is to investigate the development of the number concept of fraction in children. The study presented in this paper is part of a larger set of studies aimed at assessing children's quantitative notion of (positive) rational number (see also Wachsmuth et al., 1983).

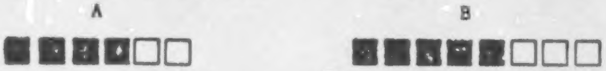
RATIO AND PROPORTION IN THE STUDIES OF NOELTING AND KARPLUS

The Orange Juice Tasks (Noelting, 1980) and Lemonade Puzzles (Karplus et al., 1980) were experiments aimed at assessing the development of proportional reasoning in children and early adolescents. Noelting's study differentiated developmental stages for subjects from ages 6-16 years with respect to problem types that he hypothesized to depend on the development of ideas of ratio and proportion in children. We are interested in the cited studies since a subject's ability to deal with a proportional situation might be an indicator for the quantitative concept s/he has developed of the ratios involved. The concentration of a mixture resulting from, say, 4 parts orange juice and 2 parts water in some sense embodies the quantitative aspect of the ratio 4:2 -- it is a concentration of "4/6 orangy" (i.e. 4 parts sirup per 6 parts liquid; note that only a transformation of the part-part ratio to the part-whole ratio will yield a measure for the concentration). The comparison of the concentrations, for example, of a 4:2 and a 2:1 mixture requires that children realize that these ratios, though different in quantity, have the same value (are equivalent). At Stage II A in Noelting's hierarchy (1980) children begin to realize that there is an internal relation between the two terms of a ratio (or between numerator and denominator of a fraction) whose value is independent of the total quantity of liquid. Consequently, an assessment of children's performance on mixture tasks could elicit to what an extent they employ strategies that are based on the "within relation" (Noelting) of terms, in other words, exhibit children's quantitative understanding of ratio or fraction.

CAN MIXTURE TASKS GIVE INSIGHTS INTO SIZE PERCEPTION?

Several reasons suggest an ambiguity over whether children's performance on the quoted mixture tasks adequately documents their understanding of proportion and rational-number ideas. (1) According to Karplus et al. (1980, p. 141) there is evidence that "consistent use of proportional reasoning is not a developmental outcome, but depends instead on overcoming task related obstacles." (2) Noelting and Gagné (1980) contrasted the orange juice tasks with the "Sharing Cookies" experiment which suggests the comparison of fractions rather than ratios, and with comparisons of numerical fractions, with identical numbers from one experiment to the other. They found low correlation between subjects, and differences between these and the ratio situations which they explain "by the greater importance of 'between' relations in ratio and of 'within' relations in fraction" (p. 132). This suggests that children's performance might depend on the problem representation. (3) Comparative items (i.e., "which of two mixtures is stronger?") do not explicitely assess children's perception of the size of a single fraction or ratio. (4) It is unclear to what an extent a child's reasoning is based on visual perception rather than perception of the size of the involved ratios. Perhaps the failed answer of one of Noelting's subjects (1980 , p. 225): "Because there is less water in A (4:2) than in B (5:3)" was based on the right impression, namely, that there is relatively less water in A than in B (Figure 1 suggests that there are water parts for only half of the orange parts in A but for more than half of them in B).

Figure 1



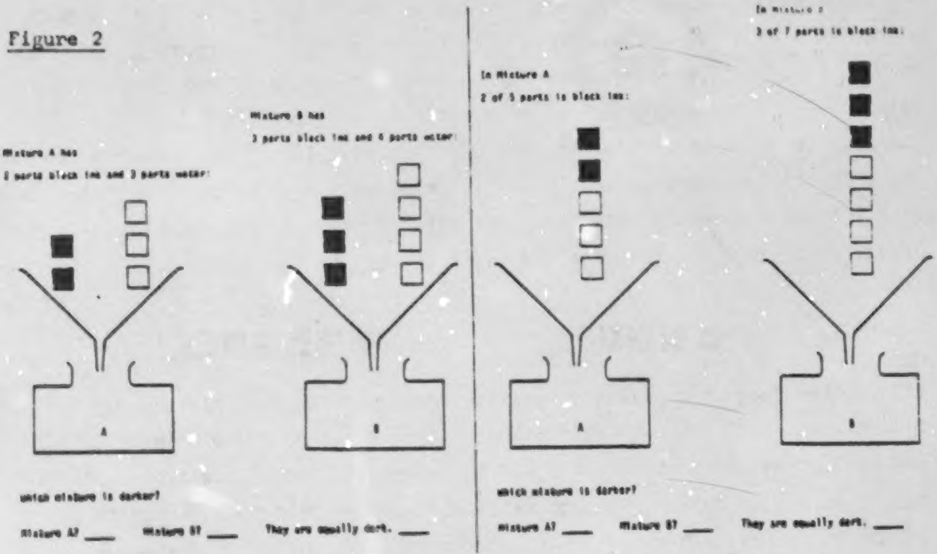
(5) Another subject's answer to the item just mentioned: "A gives two wholes and B: 1 2/3" was passed, but is based on a wrong perception of the concentrations (two wholes is not the concentration of the orange mixture but it is 4/6 of 2/3). Although mathematically correct in a comparative situation, since $a/b < c/d \Leftrightarrow a/a+b < c/c+d$, the subject's reasoning presumably was not based on this insight. (6) In general, pictures as in Figure 1 can be interpreted in various ways: as fractions (i.e., 4/6 vs 5/8), or as ratios (i.e., 4:2 vs 5:3, with the possibility of reading them as four-halves and five-thirds). Thus, we cannot be sure to what an extent observations of children's perception of the size of ratios and fractions in mixture tasks are distorted by task-related variables.

THE INK-MIXTURE AND THE GRAY-LEVELS STUDIES

In video-taped clinical interview settings close to and at the end of the 30-week teaching experiment in the Rational Number Project, two studies were conducted to get further insights into children's quantitative understanding of ratios and fractions. Subjects were 16 fifth-graders, eight from each of two experimental groups in elementary schools in DeKalb, Illinois, and a suburb of Minneapolis, Minnesota.

The ink-mixture study was done after 27 weeks of instruction. Each of 9 separate tasks was concerned with the comparative darkness of two ink mixtures, similar to the mixture tasks of Noelting and Karplus, but was presented twice contrasting a "ratio format" with a "fraction format": Having different pictures and wordings, one version suggested a part-to-part while the other suggested a part-whole interpretation of the same problem (see Figure 2). Before answering the questions, subjects had been shown possible results of mixing black ink and water in different ratios in form of gray-colored cards. One card was pointed out to represent a mixture where "2 parts is black ink

Figure 2


Mixture A has
2 parts black ink and 3 parts water:
Mixture B has
3 parts black ink and 4 parts water:
A
B
Which mixture is darker?
Mixture A? ___ Mixture B? ___ They are equally dark. ___
In Mixture A
2 of 5 parts is black ink:
In Mixture B
3 of 7 parts is black ink:
A
B
Which mixture is darker?
Mixture A? ___ Mixture B? ___ They are equally dark. ___

and 2 parts is water" (or, "2 of 4 parts is black ink," respectively) and named "half-way between clear and black." Subjects were then to rate three ratios (fractions) in their darkness values, requiring discrimination between five different gray levels. More than 90% of all responses were correct, showing that subjects (1) understood the problem setting and (2) had a rough size notion of the ratios (fractions) involved. In both versions, the numerical relationships between the ratio (or fraction) components in the 9 tasks covered the full range of Noelting's developmental stages, with three cases where both ratio and fraction version were in the highest stage, III.

The gray-levels study was done after completing the 30 weeks of experimental instruction. The aim was to obtain a more fine-tuned record of subjects' perception of fraction and ratio size than the one given with the gray-color-cards experiment preceding the ink mixture study. Embedded in an ink-mixing situation, gray-color cards were to be associated with the values of fractions with a scale of 11 distinct gray levels increasing in darkness from 0% (white) to 100% (black) in stages of 10%. Twelve cards with fraction symbols representing ink mixtures were to be ordered from lightest to darkest and, based on their darkness values, to be associated with gray levels in the scale (see Figure 3). In the parallel version, 12 cards with ratio symbols were used representing ink mixtures of the same darkness values as the fraction cards (i.e., 4:2 in place of 4/6, etc.).

Figure 3

RESULTS OF THE INK MIXTURE STUDY

(a) Assessment of ratio comparison

In the presentation of results, items that meet Noelting's stage III (i.e., general non-equivalent ratios) are distinguished from those below Stage III (i.e., equivalent ratios and ratios that either have equal first or equal second components). Response explanations were categorized to discriminate between perceptually-based responses, responses which used ratios and proportional reasoning, and responses which used fraction thinking on a ratio task. In *no* case was the comparison of ratios based on the corresponding part-whole fractions expressing the concentration of ink in water (e.g., 4/6 for 4:2, etc.).

From the items that represented stages *below III*,72% (69) of all (96) responses were passed; among these: 46% (32) of all passed responses were based on consideration of ratios, 32% (22) of all passed responses were based on visual perception, 22% (15) of all passed responses were based on ratios read as fractions (e.g., 4/2 for 4:2, i.e. did no longer deal with the actual concentrations).

From the items represented *Stage III*, 33% (16) of all (48) responses were passed; among these: 25%(4) of all passed responses were based on consideration of ratios, 44% (7) of all passed responses were based on visual perception, 31% (5) of all passed responses were based on ratios read as fractions. Besides, 28% (9) of all *failed* responses (19% of all responses) to Stage-III items were rated as "perceptually based, right answer, wrong explanation."

(b) Assessment of fraction comparison

From the items representing stages *below III*, 58% (46) of all (80) responses were passed; among these: 61% (28) of all passed responses were based on consideration of fractions, 24% (11) of all passed responses were based on visual perception, 15% (7) of all passed responses were based on the corresponding part-part ratio (e.g., 4:2 for 4/6). In one case an item was failed in the category "perceptually based, right answer, wrong explanation."

From the items representing *Stage III*, 30% (19) of all (64) responses were passed; among these: 68% (13) of all passed responses were based on consideration of fractions, 21% (4) of all passed responses were based on visual perception, 11% (2) of all passed responses were based on the corresponding part-part ratio. Besides, 24% (11) of all *failed* responses (17% of all responses) to Stage-III items were rated as "perceptually based, right answer, wrong explanation."

Observations

Overall, performance on ratio comparisons was better than on fraction comparisons (72% vs 58% success frequency for below-Stage-III items and 33% vs 30% for Stage-III items). In both formats, about twice as many responses of the (fifth-grade) subjects were passed on below-Stage-III items than on Stage-III items. In both formats, about one-fourth of all failed responses with right answers had wrong explanations which indicated that the answer was based on perception. This suggests that the pictorial presentation of mixture items helps subjects to give a right answer which they might not have obtained in a purely symbolical problem setting. In the fraction format, more than 60% of passed responses on items of all stages were based on fraction reasoning. This is in contrast to the ratio fromat: In particular for Stage-III items, only one-fourth of passed responses were based on ratios representing the true darkness value of ink mixtures while for nearly half of all responses an explanation was given which was based on visual perception. About one-third of all passed responses on Stage-III ratio items was based on fractions that did not represent the true ink concentration (or, the true value of the original ratio). Though mathematically legitimate for comparison situations these answers do not contribute to insights about children's quantitative understanding of ratio.

Results of the gray-levels study, as well as further comments on the ink mixture study, will be presented in the session.

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THE DEVELOPMENT OF RATIONAL NUMBER CONCEPTS

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Fractional number ideas historically have been and continue to be difficult for students in mathematics. Yet, fractional numbers are critically important because they represent a means by which a person can deal numerically with continuous phenomena in the same way counting ideas can be applied to discrete phenomena.

One reason for this difficulty is the complexity of fractional numbers. Kieren (1979) has posited the existence of five sub-constructs which appear necessary for a fully functional fractional number construct. These are part-whole, ratio, quotient, measure, and operator. There is still no clarity as to how these sub-constructs develop, though previous research by Noelting (1978, 1980), Kieren and Southwell (1979), Karplus (1980), Noelting and Gagné (1980) and Southwell (1980, 1981, and 1982) on the operator and ratio sub-constructs have given some indications.

In order to develop a fully functional construct of rational number, the development of each sub-construct needs to be examined. Does the quality of each sub-construct change with age? If so, how does each growth point compare across the sub-constructs? How do observable mechanisms used affect the growth of sub-constructs? How do the representations used affect the student's learning? What curricular approach will be most effective for students at different levels and in different situations?

PREVIOUS STUDIES

The study by Kieren and Southwell (1978) resulted in the following conclusions in relation to the operator sub-construct:

(i) A subject who mastered the direct fractional number task was likely to master the related inverse task. It appears that it is the nature of the operator that determines performance rather than the aspect of inverse within an operator.

(ii) A pattern of growth over time is indicated in the operator sub-construct of rational number. Three major levels of development seem to occur, the "½" operator level, the unit operator level and the general operator level.

(iii) The difference between male and female performance on the machine mode was found to be significant at the 0.01 level.

(iv) Children in the elementary school appear to be able to handle composition of two functions provided they already have mastery of the two original component parts.

(v) There appears to be a delay of two years between mastery of the tasks using the two different representations, machine and pattern, the latter being a more abstract approach.

A further study by Southwell with a sample of Papua New Guinean students, in which the machine representation was again used, as well as the orange juice test (Noelting, 1978) resulted in similar conclusions in relation to inverse tasks, a general pattern of achievement by age but with a greater delay than with the Canadian sample, and sex differences.

At the same time the data analysed indicate that the stages of proportional reasoning identified by Noelting (1978) do hold with the Papua New Guinean sample with some variations. The similarity between two samples appears to increase as the stages develop. Differentiation in these stages does not appear to be determined by grade.

The strategies adopted by Papua New Guineans are similar to those of Canadian subjects at the pre-operational and concrete operational levels. No clear statement of strategies within the formal operational level is possible.
A study (Southwell, 1982) using an Australian sample indicates that Australian students develop the operator sub-construct of rational number in three general stages with a delay of approximately one year on Canadian students. Findings regarding inverse functions and composition reflect the Canadian results.

THE PURPOSE OF THE STUDY

In order to extend the findings of previous studies and in pursuance of an ultimate goal of determining the most effective approach to the teaching of fractional numbers, a further study was undertaken. The purpose of this most recent study was:

(i) to trace the development of the various sub-constructs with grade,

(ii) to compare performance on different modes of representation of the rational number sub-constructs

(iii) to analyse the mechanisms used by subjects in handling fractional number situations, and

(iv) to examine implications for teaching rational numbers.

SAMPLE

The sample consisted of 522 subjects from Grades 6, 8 and 10 in four schools in Lae, Papua New Guinea. The secondary subjects were from two different Provincial High Schools, both of which are mainly boarding schools. The sample distribution is given in Table 1.

Table 1
Sample Distribution

	Grades			
Sex	6	8	10	Totals
Female	76	115	67	258
Male	81	101	82	264
Totals	157	216	149	522

INSTRUMENTS AND PROCEDURES

The first test administered was the Rational Number Thinking Test developed by Kieren who gave the investigator permission to use the test. Because of the relative unfamiliarity of Papua New Guinea students with pizzas, this was changed to "pie".

Kieren's test falls into four sections:

Items 1 to 6 have to do with mixing chocolate drink and is a form of Noelting's orange juice test. Items 7 to 14 involve sharing of pizzas (pies), giving a different representation of proportional reasoning. Items 15–18 involve the operator sub–construct and is a pencil and paper version of the concrete machine mode used by Kieren and Southwell previously. The remaining items 19–24, present various problem situations in which the subjects are asked to perform rational number tasks. These emphasised the part–whole and quotient sub–constructs.

The second test used was a much shorter test, devised by the investigator to test the subjects' acquisition of the measurement sub–construct. Items 1 and 2 involve the division of a length into a number of equal parts and the designation of each division point by some fractional name. Items 3 and 4 involve the comparison of two fractions, while items 5 to 8 test the marking and reading of scales.

RESULTS

(i) Achievement By Grade

To trace the development of the various sub–constructs by grade, the means for each sex in each grade for each sub–test were computed. These are shown in Table 2.

On the Drinks, Share, Operator, Inverse and Scale sub–tests, there appears to be a gain in achievement by grade for both males and females. This pattern does not occur for males in grades 6 and 8 on the Problems sub–test nor with males and females on the measurement sub–test. Females in grade 8 and 10 on the inequalities test do not follow the pattern either.

Table 2
Means on Eight Sub-tests by Six and Grade

Sub-tests

Grade	Sex	Drinks	Share	Operator	Inverse	Problems	Meas-urement	Inequ-alities	Scale	Total
6	M	2.16	2.88	.11	.01	2.06	2.88	.18	.06	5.704
	F	1.85	2.47	.17	.03	2.17	3.12	.15	.04	6.1852
8	M	2.20	3.67	.63	.21	1.92	4.01	1.09	.59	6.2667
	F	2.27	3.01	.72	.17	2.45	4.66	1.08	.40	6.5625
10	M	2.82	4.23	1.5	.54	2.71	3.17	1.21	0.85	7.000
	F	2.42	3.90	.97	.27	3.00	2.67	0.90	0.79	5.8750
Total	M	2.38	3.60	.74	.25	2.21	3.40	.84	.51	5.9778
	F	2.20	3.03	.63	.16	2.50	3.70	.76	.40	6.2549
Total		2.29	3.34	.69	.203	2.35	3.57	.80	.45	6.1250

An SPSS crosstabs analysis, however, revealed significant difference between grade and male and female performance on some sub-tests. This is shown in Table 3.

Table 3
Crosstabulation of Sub-tests by Grade Controlled by Sex

Sub-tests

	Drinks	Share	Operator	Inverse	Problems	Measurement	Inequalities	Scale
Male								
x^2	17.66	43.21	10.47	1.01	19.44	65.64	23.09	6.82
df.	10	14	8	6	10	14	4	10
sig.	.06	.0001	.2336	.985	.0350	.0000	.0001	.7426
Female								
x^2	22.25	52.07	2.45	2.49	18.39	44.296	9.11	3.83
df.	10	14	6	4	10	14	2	8
sig.	.01	.0000	.8746	.6469	.0488	.0001	.0105	.8723

This indicates that the differences in mean performances between grades shown in Table 3 are not significant for both males and females on the operator, inverse and scale sub-tests.

Tests of significance show that the differences between the total means of grades 6 and 8 and grades 6 and 10 are highly significant, while the difference between the total means of grades 8 and 10 is also significant, though less so.

The significance of the gain between grades 6 and 8, however, needs to be treated with some caution since, in Papua New Guinea, only 60% of Grade 6 students proceed to secondary school. Grade 8 students, therefore, may well be of a higher standard because of the selective processes involved.

(ii) Comparison of performance on different modes of representation

A correlation analysis was carried out to measure the strength of the relationship between the eight sub-tests. Significances are shown in Table 4.

Table 4
Significances of Correlations between Sub-tests

	Drinks	Share	Operator	Inverse	Problems	Measurement	Inequalities	Scale
Drinks	—							
Share	.001	—						
Operator	.001	.001	—					
Inverse	.001	.001	.001	—				
Problems	.001	.001	.001	.019	—			
Measurement	.054	.293	.007	.046	.001	—		
Inequalities	.001	.001	.001	.001	.001	.001	—	
Scale	.055	.016	.001	.001	.003	.004	.001	—

The relationship between most sub-tests seems to be strong with the exception of the measurement and share sub-tests. Analysis of correlations by sex and grade does not yield such clear significances.

(iii) Sex Differences

The previous result is supported by the fact that there were no significant differences between the total scores for males and females.

Further analysis is needed to ascertain differences in performance on each sub-test

DISCUSSION AND INITIAL CONCLUSIONS

As this study was commenced in February, 1983, final results are not available at the time of writing. A comparison of the level at which mastery is achieved on each sub-construct has not been completed, nor has a detailed investigation of mechanisms used by the subjects. Until these are completed, no implications for teaching fractional numbers can be considered.

From the data presented, however, it seems that there is an increase in performance by grade, though no information is yet available as to the precise nature of that increase. Also, there appears to be a strong correlation between performance in the sub-constructs measured by the tests administered.

NOTE: Detailed reports of this study will appear in the Mathematics Education Centre Report series of the Papua New Guinea University of Technology, Lae.

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HOW CHILDREN ACCOUNT FOR FRACTION EQUIVALENCE[1]

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Knowledge of fraction equivalence is necessary for a mature understanding of rational number. If our task as mathematics teachers is to "reduce the degree to which students view mathematics as a bag of tricks containing several "magical" procedures which miraculously produce answers in narrowly defined situations" (Post, 1981, p30), then we need to teach our students meaningful bases for thinking about fraction equivalence. Observation of student problem solving behaviour is a sensible source of such information. Research conducted recently in Australia, Europe and the United States shows that the key concept of fraction equivalence is not learned well in Western school systems.

Nine Year 4, 10 Year 6, and 10 Year 8 students from two elementary schools and one middle school in Clarke and Walton Counties, Georgia, were interviewed in the spring of 1979. Sets of problems designed to expose students' thought processes in discrete quantity partition contexts, and fractional number contexts were used (Hunting, 1981). Data from students' responses to Task F9: Solving Equivalence Expressions is reported here. For this task, students were individually shown a fraction written on a sheet of paper, and asked to complete an equivalent fraction whose denominator was given. Then, using a supply of counters, they were asked to verify that the two fractions were equivalent. Problems given included 1/2 = /4; 1/4 = /8; 1/4 = /12; 2/6 = /3; 6/15 = /5; and 3/12 = /8. An adaptation of the clinical method was used to obtain the data. Of interest in this study were the kinds of strategies used to produce and to verify solutions to the equivalence problems. Production strategies will be described first, followed by an example of verification behaviour from the first equivalence problem, 1/2 = /4.

RESULTS

Types of production strategies displayed by students in completing the equivalence expressions in Task F9 were these:

1. Common factorisation. In this strategy arithmetic procedures

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[1] An expanded version of this paper is available upon request.

directed towards the identification of a common factor in both numerator and denominator were employed. For example, MK(12;1) reported solving the problem 2/6 = /3 this way: "In order to get that, you had to divide six by two times, and so you had to divide two by two and you got one." This strategy worked for problems where one denominator was a multiple of the other.

2. Cross-multiplication. For the problem 3/12 = /8, KB (12;3) multiplied 3 by 8 to get 24, then divided 24 by 12 to find the required numerator.
3. Recalled knowledge. LL(9;11) stated that she knew two-fourths was equivalent to one half because she remembered this fact from the previous year's instruction.
4. Invented algorithm. With the problem 2/6 = /3, DD(9;10) multiplied the numerator of the fraction 2/6 with its denominator, and obtained 12. He then found that numerator which, when multiplied by three, would also give 12.
5. Use of ratios. CP(10;3) explained his solution to 1/4 = /8 by saying that since the denominator four was increased by four to make eight, the numerator one should increase by one to make two.
6. Intermediate fraction. LN(14;3) successfully completed the expression 1/4 = /8 saying: "I figured four-eighths is one-half and one-fourth is one-half of one-half."
7. Guess and see. Students would typically offer a solution in the expectation of receiving further information from the investigator. For example, SR(10;0) chose three as the number of fourths equivalent to one-half.

I : "Why did you decide to write three?"
SR : "Because two plus one is three I guess."
I : "O.K."
SR : "I know that's not right."

Occurrences of these strategies across all the children interviewed can be seen in Table 1. Not surprisingly, the older children completed more problems. Several Year 4 students who did not respond, or guessed solutions to the first problem, 1/2 = /4, were able to successfully produce solutions by constructing representations for the fraction 2/4 using the available counters. Since such physical experiment was not a spontaneous response to the problem given it was not categorised as a production strategy. Overall, the most popular strategy (46%) was

YEAR 4 STUDENTS (N = 9)

Equivalence Problems	AD(9;11)	BH(10;4)	CP(10;3)	DD(9;10)	EF(10;2)	MP(9;8)	LL(9;11)	MC(10;5)	SR(10;0)
1/2 = /4	NR	7	NE	4	NR	7	3	1	7
1/4 = /8	NE	NE	5	4	-	7	4	1	7
1/4 = /12	4	-	NE	6	-	-	NR	-	4
2/6 = /3	-	-	4	4	-	-	4	-	4
6/15 = /5	-	-	-	-	-	-	-	-	-
3/12 = /8	-	-	-	-	-	-	-	-	-

YEAR 6 STUDENTS (N =10)

Equivalence Problems	KB(12;3)	TB(12;5)	CC(11;8)	MK(12;1)	KL(11;10)	TM(12;9)	AR(11;5)	TR(13;4)	DS(11;10)	MW(11;9)
1/2 = /4	2	4	1	1	5	1	1	EO	4	1
1/4 = /8	2	4	NE	1	NE	1	1	EO	4	1
How many fractions equivalent to 1/4?	-	-	You could go on and on. There's probably no telling	I don't know. There'd be an awful lot.	About 10 of them	About 8	There's a lot of them	-	-	Could go on and on.
2/6 = /3	1	NR	1	1	5	1	NE	4	4	1
6/15 = /5	1	1	1	NE	4	4	1	4	4	1
3/12 = /8	2	NR	4	NR	NR	4	NR	NR	-	NR

YEAR 8 STUDENTS (N =10)

Equivalence Problems	CB(13;5)	JB(14;7)	MG(13;10)	SH(14;8)	LN(14;3)	AO(14;1)	JP(13;8)	MS(14;4)	DW(13;11)	VW(14;6)
1/2 = /4	5	1	1	1	3	5	1	1	1	5
1/4 = /8	4	1	NE	1	6	5	1	1	1	1
How many fractions equivalent to 1/4?	You can go on and on	I don't about 10 or 12	You can keep on going as long as you want to	All of them	They would be infinite because numbers are infinite	A whole lot of them	An infinite number	-	A lot	Since there's no end in numbers, you could most probably four times any number
2/6 = /3	1	1	1	NE	1	5	1	NE	4	1
6/15 = /5	NE	1	1	4	NE	-	NE	1	4	1
3/12 = /8	4	4	6	2	4	-	4	NR	4	1

Key:
1. common factorisation
2. cross multiplication
3. recalled knowledge
4. invented algorithm
5. use of ratios
6. intermediate fraction
7. guess and see

-	problem not given
NR	no response
NE	no explanation
EO	explanation obscure

TABLE 1 : PRODUCTION STRATEGIES

common factorisation, followed by invented algorithm (31%). Frequencies of other strategies were: use of ratios (8%); guess and see (5%); cross multiplication (4%); intermediate fraction (3%); and recalled knowledge (2%). In the second most frequently occurring strategy, invented algorithm, several students appeared to attempt common factorisation, but were unsuccessful. For example, TM(12;9) explained her solution to 2/6 = /3: "Cause you can reduce two-sixths into one-third." But in the following problem, 6/15 = /5, she wrote three because: "I said three can go into six and 15."

Many children were prevented from verifying solutions because of their inability to represent a fraction using a given set of counters. Moreover, very little evidence was found for a connection in the children's minds between strategies used to produce equivalent fractions, and strategies for verifying these expressions. An excerpt from a working transcript of a Year 8 student is provided as an example.

MG (13,10) reported using common factorisation to find the solution to the problem 1/2 = /4. He used a pie analogy to explain the equivalence of one-half and two-fourths:

> "Well, because one-half, let's say you had a pie, and one-half of a pie (draws a circle and marks in a diameter) like this, this is one-half. Make it two-fourths (draws another diameter orthogonal to the first), two of those fourths make it the same as one-half. Do you understand?"

When asked if he could explain with counters MG took four counters and offered a similar argument as before. But doing so with 12 counters proved difficult for MG. He placed the counters in a 6 x 2 array.

I : "Right"
MG : That's one-half there (takes a column of six and rearranges them into a 2 x 3 array), that's two-fourths - I don't understand what you're talking about."
I : "Can you show me one-half of 12 squares?"
MG : "One-half of 12 is what. Oh, O.K. (counts out six counters). That there."
I : "O.K. Now can you show me two-fourths of the 12 squares?"
MG : "Two-fourths of the 12 squares. Right there (pointing to the six counters arranged for one-half)."
I : "How is that two-fourths?"

MG : "O.K."

I : "It looks like one-half to me."

MG : "But see, two-fourths will go into 12 (picks up pen and writes 2/4) four will go into 12 three times, and that will give you six. (writes /12 and then six over the numeral 12). This is six over here and that will give you the same thing.

I : "Yeah. Why don't you make those 12 squares into fourths for me."

MG : Into four?"

I : Into fourths."

MG : (Makes three groups of four) "Here."

I : Is that fourths?"

MG : "Yeah."

I : "How many equal groups are there?"

MG : "Three"

I : "Yeah (pause) What does one-fourth mean?"

MG : "Huh?"

I : "What does one-fourth mean?"

MG : "Oh, one-fourth of 12? It would be that right there (indicating group of four)."

MG used a common factor strategy to complete the given equivalence expression. His justification rested on a continuous quantity object: that of a pie. MG could take four discrete elements and provide an analogous rationale. But 12 counters revealed the brittleness of MG's conception of fourths. He resorted to numerical algorithms to argue the equivalence of one-half and two-fourths of 12 objects. When pressed to show two-fourths with discrete elements MG seemed unable to bring to bear a satisfactory action strategy. MG's knowledge of fraction equivalence in this context was not adequate.

DISCUSSION

Fraction equivalence is a key idea in elementary mathematics. Data provided in this paper shows that students use a variety of strategies for producing solutions to equivalent fraction problems. Surprisingly, the most universal procedure for solving the problems given, cross-multiplication, was used in only 4% of the strategies classified.

Discontinuities were observed between strategies students used for producing solutions and supporting knowledge grounded in physical reality. Even the most successful students interviewed adopted different procedures for obtaining solutions and verifying their results. A number of students were prevented from successfully demonstrating equivalence between physical representations of two fractions because they could not represent individual fractions using the discrete quantities available. The example given highlights the dependence of equivalence knowledge on possessing general action strategies for constructing physical representations for fractions. There is an urgent need for mathematics teachers to reconsider their methods of teaching fractions in the light of these results. In particular, when teaching fraction equivalence students should be expected to verify their solutions to equivalence problems using physical materials. If students experience difficulty finding equivalent fractions, then inexpensive resources such as counters, blocks, popsticks, and the like can be used to find solutions. Teachers should ensure that representations of fractions are constructed from sets whose cardinality is some convenient multiple of the denominators under consideration so that literal interpretations of fractions, like, for example, one-fourth means "one out of four things," can be broadened and extended.

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THE LONG TERM LEARNING PROCESS FOR RATIO*

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SUMMARY

In the proper contribution some remarks are made on the manifestations of ratio and proportion in actual mathematics programmes.

Next some global results of the consultation of an international panel of experts will be reported with respect to their judgements about the way one should deal with ratio and proportion in mathematics education. To serve that aim an instructional text was submitted to the panel.

In the closing sections the author reflects on some important features concerning the long term learning process for ratio, to wit (on ratio) anticipatory activities, modeling and schematising.

1) INTRODUCTION

In mathematics programmes for elementary instruction, one is often struck by the poverty and brevity of the approach of ratio.

The poverty of the approach to ratio is more generally characterised by

- concept building, exercised with mathematical objects;
- virtual lack of real applications;
- isolation of the subject 'ratio', which is hardly connected to any other subject;

  The intended connection, - if there is any - is imposed *a posteriori*, conceived in an advanced stage of abstraction.
- ignorance of the visual world as a source of ratio and the lack of visualisation in the approach of ratio; for instance no exploration of the phenomenon stick-sun-shadow, which is a most efficient model for ratio invariance.
- virtual lack of schemas for the numerical processing of ratio problems;

2) 'THE GIANT'S REGARDS' FOR AN INTERNATIONAL PANEL.

The IOWO-theme 'With the giant's regards' (De Jong, 1978, p. 23-43) has been re-edited with a view on a research study. The original version was centered around area as a basis for ratio.

In the new version it has been tried to approach ratio in a more variegated way.

More justice had to be done to the phenomenon 'ratio' as it appears in the world of and for children. It had been my intention to submit the theme to an international panel of expert colleagues in order to learn what they think about ratio and to confront their views with my own. The group that had been invited, consisted of curriculum developers, teacher trainers, researchers (psychologists and educationalists), mathematicians, and subject matter didacticians (mathematics, physics, chemistry).

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*A more extended paper will be available at the conference.

They numbered 68. (40 from abroad, 28 from my country). 32 reacted (from each group 16), and 29 responses could be used for the present research.

A text containing a description of the theme, close to actual instruction and illustrated by pictures and drawings had been submitted to the panel. A lesson plan had been added, from which the various objectives could be globally derived.

For instance:

- ratio and measuring length (lesson 1);
- measuring and increasing in two directions (lesson 5).

These objectives had been specified by descriptions of lessons and suggestions for mathematical activities.

The panel was informed that the lessons were meant for third graders (8-9 years old). The working sheets for the pupils were part of the material submitted to the panel.

The panel were asked questions like: 'Do you judge that in the theme' 'With the giant's regards'' full justice has been done to the phenomenon 'ratio'? and 'Does ratio in all its aspects occur in the proposed activities?'

3) MAIN TRENDS IN THE RESPONSE RECEIVED

A quarter of the respondents who explicitly dealt with the first question, judged that justice had been done to the phenomenon 'ratio' albeit within a restricted context.

It may be concluded that the embodiment of mathematical activities into contexts was judged to be essential for learning mathematics with a view on a many sided approach (and embedding) of ratio and the applicability of the learned subject matter.

In general one might conclude that the views brought forward in the responses of the panel were as it were the opposite of each of the characteristics of actual mathematics programmes as mentioned in our introduction.

Indeed they stressed such aims as the following:

- respecting the aspects and manifestations of ratio; comparing all kinds of magnitudes; considering mixtures of continuous quantities; distinguishing *internal* and *external* ratios (that is ratios *within* and *between* magnitudes; cp. Freudenthal, 1983, ch. 6) stressing ratio in the operator;

- providing ratio a meaning by problem situations in a context, that is to say the use of reality both as *a source* (for concept building) and *domain of application;*
- recommending the use of schemas and visual models to support the learning process;

According to the opinion of the panel the necessity of long term learning processes was beyond doubt. It was characterised by such qualities as the following:

- from intuitive notions via abstracting to concepts;
- applicability of the learned subject-matter;
- many sided approach to ratio;
- consciousness about the learned subject matter acquired among others by conflicts and reflection.

In the long term learning process the mental status of 'ratio' moves from intuitive notion to fullfledged concept. A primarily qualitative approach and activities of estimating can play an important part. (cp. Streefland, 1982).

In the plea for coherence in the curriculum,fraction and similarity were mentioned as obvious examples.
The long term learning process played a part in many responses though it was neither elucidated by arguments of content nor by examples.
By means of an analysis of some phenomenon, - which can only be referred to in a general way within this framework - and with a view on activities that are related to, or anticipate on ratio, much more subtle connections can be shown within the curriculum.

4) ACTIVITIES ANTICIPATING ON RATIO AND THE FIRM EMBEDDING OF RATIO IN THE CURRICULUM

In our view the *keypoint* for a solid embedding in the curriculum is the mathematical material *on which ratio is based:*
In order to explain for the very moment what I mean, I consider the example of population density.
In this 'composite magnitude' a result of *counting* is connected with a result of *measuring.*
The evolving construct is that of equipartition of inhabitants per unit of area in order to facilitate comparing such a situation with similar ones: more or less populated areas.
In general taking the stand of 'ratio' means relativating the results of what might be called basical mathematical or physical operations such as measuring, counting, estimating, reckoning and their composites.

The *logical status of ratio* is more complex than that of such elementary ideas as length, mass, area, volume, number, adding, substracting, multiplying, dividing (cp. Freudenthal, 1983; p. 179. 181; Streefland, 1982, p. 194).

Examples, to be elaborated in detail at the conference are *density, multiplication and probability.*

'Counting large quantities' and 'measuring areas' - among others - are *together* the very cognitive and numerical sources of ratio.

The various separate learning sequences meet each other in the learning sequence for ratio. It is a key question in which stage the separate learning processes, should be *intertwined* to prepare ratio. Or - seen from the viewpoint of ratio -: How early are *informal approaches* to functional bonds between magnitudes to be explored in order in a later stage of the learning process to be recognised as similarities, linear mappings, linear functions, and so on.

Objections against too late intertwining and forced formalising might be derived from research results. (preposterous abstraction, premature algorithmisation; cp. Hart, 1980, 1981). What is proposed here is intertwining of such learning sequences in a quite early stage while linking up with children's intuitive, informal solving methods rather than forcing up algorithms and badly fitting models. Intertwining should attribute to the constitution of mental objects within the particular learning sequences as well as to that of ratio as its product.

The kind of activities at junctions of learning sequences will be called *anticipating:*

This choice requires more complicated problem situations. In particular with respect to ratio this greater *conceptual complexity* is indispensable. (Vergnaud, 1982, p. 53).

As the learning process progresses the aim of anticipating activities moves on too.

The long term learning process was implicit to many reactions I received, although such considerations as brought forward in the present subsection were at most globally indicated.

Sofar radical interventions in the prevailing mathematics programmes for 6-13 years olds as the proposed one have not been considered. On the contrary one has yielded to the temptation to delay ratio even more because of difficulties experienced in the teaching practice.

5) MATHEMATICAL TOOLS TO SUPPORT THE LONG TERM LEARNING PROCESS

The features of modeling and schematising played an important part in my reflection on the long term learning process.

Modeling

The models at issue are visual ones which support the learning process and broaden the applicability of the acquired knowledge. Starting point will be *model building within the learning process.*

A sequence for the sector diagram for instance, which outlines the long term learning process, might be:

- exploring and producing (cooking) recipes;
- visualising cakes in circle diagrams what regards the ingredients of the recipe and their part in the whole;
- transition from a variable to a fixed representation: the diagram is going to function as a *model*, the sector diagram;
- translating (other) recipes, mixtures, alloys, in sector diagrams (in order to facilitate questions of comparison, for instance);
- more examples of divisions and distributions put into sector diagrams, for instance:
  time tables of activities of a counter clerk (stamp sales, money matters, advising, ...) or of a policeteam (service, traffic regulations, crime fighting, ...);
- interpreting given sector diagrams, comparing and ordering them according to one or more criteria;
- applying sectordiagrams in general

Schematising

DISTANCE (m)	5	10	12½	15			
TIME (see)	1	2	2½	3			

The ratio table can play a decisive part in the learning process with respect to the development of algorithmic procedures for the comparison of ratios.

As we will show at the conference, the ratio table schematises in a well-organised way the spontaneous problem-solving strategies of children, without cutting off possibilities.

Specifying the part played by the ratio table in the long term learning process, we arrive at

- support given to the concept building (ratio as an equivalence relation, the concept of variable);
- contribution to the detachment from the context (the ratio table as a unifying model);
- contribution to discovering, making conscious and applying all properties which characterise ratio preserving mappings and to their use in numerical problems;

Finally:

- serving the process of algorithmisation of ratio (and fractions, percentage, ...) via shortcuts adapted to the solving process of problems (cp. Streefland), 1982)

6) CONCLUSION

Learning ratio should start from genuine problems in the reality of and for children as a *source* (for the acquisition of the concept) and *domain of application.*

The visual perceptive reality of children serves as a start. Justice should be done to the various manifestations of ratio.

The learning process should be designed with anticipating activities that account for connections with other learning sequences. To this aim schemas and visual models should be developed to support the long term learning process and therein general cognitive processes such as abstracting, generalising and unifying. Stress must be laid on the mathematical *activity by* which schemas, models and procedures of problem solving will be *constructed* by the children themselves in order to bring about adequate concept *building*.

Abstractions, generalisations and models are to be developed from the spontaneous informal solving strategies, rather than to be forced on the learners in order the learned subject to be made applicable.

Those are the germs of an instruction theory with respect to learning ratio and proportion.

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PROBLEMS OF REPRESENTATION OF AN OPERATION IN ELEMENTARY SCHOOL ARITHMETIC

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PROBLEM

Our previous research on the understanding of numeration in elementary school children (1) and work with children in classrooms has led us:

1) to observe that different representations of the same concept were interpreted or used by children in various ways, and often not in the expected or desire sense;
2) to the analysis of multiple forms of symbolization with which children are confronted in schools texts, tests and other curriculum materials.
3) to observe that in a learning context children construct their own representations of concepts and modify them gradually over time.

The following questions concerning the importance of representation have been posed:

1) Are children imposed prematurally to representations in mathematics that are inaccessable or not useful? In such cases they cannot react other than by ignoring them, or by the mechanical application of a rule.
2) To what degree are these representations a support for the understanding or solution of a problem?
3) What interpretations do children give to multiple forms of representation?
4) Do they see in these representations what the adult would like them to see?
5) Are there representations that are altogether useless or create learning problems?
6) What spontaneous representations do children use when a problem is presented?

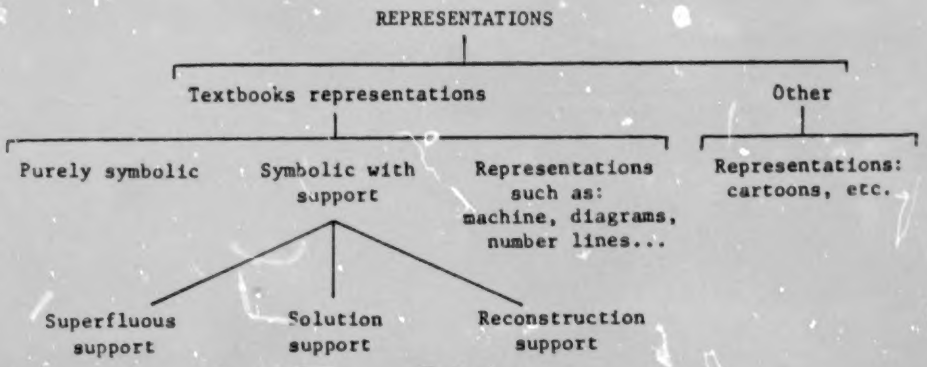
CURRENT RESEARCH

In our analysis of childrens' textbooks concerning representations related to operations in arithmetic, we have encountered symbolic representations of two general types: a) those purely symbolic (e.g. equations) and b) those which are symbolic plus another representation to serve as a support (drawings, set diagrams, number line, machines, etc.). In the case of symbolic representations with support, we distinguished three sub-categories of supports:

1) "*superfluous supports*": Those that are intended to serve as a support, but in fact bear no relation to the symbolic representation and do not help the pupils to solve the symbolic form.
2) "*solution supports*": They focus on the association expected from the child between the symbolic (equation) and the support. Despite, a total one-to-one correspondance between the symbols and the support is presented and no mental activity or solution process is promoted. It is actually short-circuited.
3) "*reconstruction supports*": They require the same strategy either on the support or on the symbolic form.

Examples will be provided at the conference. We also find forms of representation without symbolic equations, which are intended to help children see different aspects of operations: machines, set, diagrams, etc.

Moreover, we use representations which are not met in textbooks and which come out on one hand from the spontaneous creation of pupils in learning situations or on the other hand from a special search (depending on context) for representation more evocative of actions.


REPRESENTATIONS
Textbooks representations
Other
Purely symbolic
Symbolic with support
Representations such as: machine, diagrams, number lines...
Representations: cartoons, etc.
Superfluous support
Solution support
Reconstruction support

EXPERIMENTATION

Different situations were constructed, in accordance with the various questions previously identified.

Six types of situations were devised up to now:

1) Free discussion with the children concerning the different representations identified previously.
2) Symbolic equation with supports. Children resolve the equation and then, from memory are asked to describe the support they had seen. This was intended to study if in fact the children actually used the support presented.
3) Different representations were presented (drawings, machines, cartoons, etc.) and children were asked to invent a problem starting from this representation.
4) Different representations were presented and children were asked to write the corresponding equation.
5) Problems corresponding to different contexts of the four arithmetic operations (e.g. for multiplication context of repeated action and context of rate) are presented to the children and we note which supports they use in the solution. In the case of word problems relative to different contexts of operations, there are two types of problems: for certain problems children should see an action over time, for other problems children should see static states.
6) Different representations are introduced (in textbooks or otherwise) and children are asked to solve problems using this representation.

Some of the situations were experimented in the form of an interview (1, 2 and 6 above) with six to nine years old children. Other situations were experimented in written form (3, 4, 5 and 6 above) with children from six to twelve years old. In the written form, approximately fifty children participated at each grade level.

RESULTS

In the three sub-categories previously identified, we verified if the selected classifications proved adequate and identify the representations which 1) are superfluous, 2) stress more efficiently than others certain features of arithmetic operations, 3) represent actions more adequately (Can a representation on paper reproduce the dynamical nature of an operation?) and 4) are being developed by children in learning situations can be as well used by other children.

Moreover, we verified if children use the same sort of representations in various contexts: those involving actions or those more static in nature.

We hypothesized that children will not take the same type of representation in these two types of problems. Furthermore, the children have more difficulty to represent a problem where there is an action than in static contexts.

Since this is work now in progress the results of our studies will be communicated at the PME meeting in July.

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A CRITIQUE OF PIAGET'S ANALYSIS OF MULTIPLICATION

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Nowadays, teachers use many different models in their presentation of multiplication. Some textbooks introduce this operation by counting the number of jumps on the number line, others refer to the Cartesian product (number of blouses x number of skirts = number of outfits), while still others are based on problems involving the concept of ratio. This host of presentations reflects the large number of situations which can be represented by multiplication. But the question arises whether such models constitute a good intuitive basis for the initial construction of this arithmetical operation or, if on the contrary, they involve more complex notions than that of the quantification of sets of discrete objects.

Quite obviously, jumps on the number line necessarily involve the concept of measure of length which proves to be more complex than that of number used as a measure of discrete quantities. The illustration of the Cartesian product brings about a fairly abstract element of combinatorics, that of choice. Moreover, such a concretization is quite relative for in the given example, it is the number of different pairs of blouses and skirts that needs to be counted. In fact, all the possible outfits cannot exist simultaneously at the concrete level since any given outfit needs to be broken up in order to construct further pairs. As reported by Suydam and Weaver (1970), Hervey (1966) has found that the Cartesian product model was more difficult for second graders than the equal addend model. As far as using the concept of ratio as a basis for multiplication, it is not at all evident that this is valid in the case of discrete objects, nor that such an approach might not constitute an inversion of the natural order of the child's constructions. This brief analysis seems to indicate that it is still the traditional model, that is the quantification of equivalent sets of discrete objects, which represents the most primitive of all multiplicative situations.

A review of the literature on the learning of multiplication of natural numbers shows that most of the studies dealt with the multiplication algorithm involving multi-digit numbers. Regarding the multiplication of

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numbers smaller then 10, many teachers tend to focus on the memorization of number facts without asking themselves questions such as "What does it mean to understand multiplication?", "How does the child construct this arithmetical operation?". Answers to these questions require an analysis of the multiplication concept which cannot be reduced to a simple task analysis, but needs to be carried out in an epistemological perspective.

Multiplication having been taught for ages, it is rather surprising to find how little attention these questions have received. Only a few researchers have tackled them. In fact, if a teacher or any adult is asked the first of these questions, the answer is likely to be "Multiplication is repeated addition". However, such an answer reveals an emphasis on the formal definition of the arithmetical operation, a definition whose nature is far more procedural than conceptual. It seems very difficult at first sight to identify a concrete action specific to the idea of multiplication. This is in sharp contrast with division which, although defined as the inverse of multiplication, is associated with the act of sharing.

A DEFINITION OF MULTIPLICATION

Within the context of the quantification of sets of concrete objects, it seems nearly impossible to perceive a multiplicative situation independently of the idea of division. Indeed, if the latter raises the question of sharing the whole into equal parts, the relation of the parts to the whole involves multiplication. It does not seem possible to generate a multiplicative situation which could be dissociated from a prior appropriate partitioning:

> An arithmetical multiplication is an equi-distribution: m x n is m sets of n terms or n sets of m terms which correspond one to one
>
> (Piaget & Szeminska, 1941/1967, p.253)

Of course, this definition does not describe the arithmetical operation but rather that which characterizes a multiplicative situation. If at first sight it seems to define multiplication on the basis of division, one must however keep in mind that equi-distribution does not necessarily imply that all the elements of the initial set have to be used. And it is this criterion, that of a non-exhaustive distribution which can be used to distinguish a multiplicative situation from a situation involving division. In the latter case, the complete distribution of the initial set would be required.

Although Piaget and Szeminska include commutativity in their definition of multiplication, this property is far from being accepted by all primary school teachers. In an experiment dealing with the training of teachers in the analysis of mathematical concepts (Bergeron et al, 1981), many of them insisted on a unique interpretation of 6 x 9 restricting it to "six sets of nine" and mentioned that this was the only answer they would accept from their pupils. But far from being a sign of ignorance on their part, this rejection of a double interpretation (nine sets of six) was due to their desire to model a multiplicative situation. That the same answer resulted from both interpretations was not considered a good enough reason and they gave as example that "working for six hours at nine dollars per hour" was not at all the same as "working nine hours at six dollars per hour". Their single interpretation of 6 x 9 was reinforced by a natural tendency to read from left to right, a tendency observed in other arithmetical contexts by Kieran (1979).

THE CHILD'S CONSTRUCTION ACCORDING TO PIAGET

In their book La Genèse du Nombre chez l'Enfant, Piaget and Szeminska (1941/1967) devote a whole chapter to the child's construction of the multiplication concept. According to them, this construction is elaborated on the basis of one-to-one correspondances between several sets (more than two), which involves a composition of the equivalence relations, composition which eventually brings about arithmetical multiplication.

> Moreover, arithmetical multiplication being an equi-distribution, the equivalence based on one-to-one correspondances between 2 or n collections A is thus an equivalence of a multiplicative nature whose meaning is that one of these collections A is multiplied by 2 or by n; $A \longleftrightarrow A \ldots$ thus means 2A or nA, just as conversely, nA implies the term-by-term correspondance between n collections. From a psychological point of view, this simply means that setting up a one-to-one correspondance is an implicit multiplication: hence, such a correspondance established between several collections, and not only between two of them, will sooner or later lead the subject to become aware of this multiplication and establish it as an explicit operation. (Piaget & Szeminska, 1967, p.262)

Piaget and Szeminska have described the construction of multiplication on the basis of the following experiment:
The child is asked to construct first a set of red flowers (R) and then a set of blue flowers (B), both sets to be equivalent to a given set of ten vases (V). He is then questioned about the equivalence of sets R and B. Following this, he is asked how many flowers will be in each vase if all the flowers are distributed evenly in the vases. Finally, in order to verify if the child perceives the number of flowers as being twice the number of vases, he is asked to fetch enough tubes for all the flowers (each tube containing but one flower).

By interviewing children aged four to seven, the authors of this experiment found that the perception of this situation as being multiplicative depended on a conservation of the equivalence of the sets based on one-to-one correspondence. Indeed, the child who held on to the equivalence of the two sets of flowers in spite of changes in their configuration, could spontaneously generalize this equivalence to more than two sets. Moreover, he could predict the number of flowers in each vase and, in contrast with the non-conserver, he could fetch without hesitation the quantity of needed tubes (10 and 10) thus indicating his awareness that the number of required tubes was twice the number of vases. He could also extend this to other multiples of 10.

Piaget and Szeminska use the term "multiple correspondence" to describe the generalization to n sets of the composition of equivalences based on one-to-one correspondences. Perhaps the degree of sophistication of "multiple correspondence" may be overlooked unless one recalls that according to them, composition of one-to-one correspondences also implies conservation of this equivalence. These authors claim that such a "multiple correspondence" leads to both the one-to-many relation as well as to the multiplication schema. But is this necessarily so? Are there not easier alternative constructions?

In their experiment, the one-to-many relation is based on the repetition of the one-to-one correspondence. It is only when the child constructs two equivalent sets of red and blue flowers with respect to the 10 vases, that the question about the relation "two flowers in each vase" is raised. But could not the one-to-many relation prove to be more primitive than multiple correspondence?

If this was the case, the one-to-many relation would not be stemming from a repetition of one-to-one correspondences. In fact, well before he conserves one-to-one correspondences, a five-year-old child might not only recognize equivalent sets but also generate them. This could be verified with some very simple tasks involving the selection of pictures in which "each cat has the same number of kittens" or in "making five packages of four cards in each package". This would imply that the one-to-many relation built on "multiple correspondence" is far more advanced than the one built on numerosity.

Rather than describing the child's construction of multiplication, Piaget and Szeminska's explanation based on "multiple correspondence" could be more a reflection of the constraints inherent to the tasks selected for their experiment. Indeed, the equi-distribution of flowers in the vases requires initially one-to-one correspondences from which the one-to-two relation and the notion of "twice" must be derived. On the other hand, as indicated in the example "five packages of four cards", one can generate a multiplicative situation on the basis of equivalent groupings. In such a case, the multiplicative nature of the task does not stem from a repetition of one-to-one correspondences but from a repetition of the one-to-many relation implied in the generation of equivalent sets.

There are some indications that the repetition of a one-to-many relation is a more primitive construction than "multiple correspondence". The first evidence is based on a simple task of equi-distribution. When children are given a set of more than 12 cards and asked "Can you make four packages of three cards in each package?", very rarely can one find subjects who will use a one-to-one correspondence by constructing simultaneously the four packages, one card at a time. If the first twelve cards were numbered from 1 to 12, such a distribution would follow the pattern
1,5,9 2,6,10 3,7,11 4,8,12. Instead, the great majority of children will generate these packages one at a time:
1,2,3 4,5,6 7,8,9 10,11,12 indicating that the construction of this multiplicative situation is based on the repetition of the one-to-many relation. "A package of three cards" is iterated four times.

A second indication can be found in the results of prior research. Gunderson (1953) and Zweng (1964), as reported by Suydam and Weaver (1970), have found in studying division that problems in which the number of parts was given and the number of elements in each part had to be found proved to be more difficult than problems in which the number of elements per part was given and the number of parts had to be found. In this latter class of problems the child withdraws from the initial set one part at a time and then counts the number of parts. In the other class of problems, the child does not know ahead of time how many elements are to be selected for each part and thus has no choice but to fall back on a distribution of the elements one at a time for each part, that is using a one-to-one correspondence repeatedly. A plausible explanation of this difference in the level of difficulty between the two types of problems is that one class of problems can be solved by using the "one-to-many" relation in the selection of the parts, whereas the other class of problems requires the repeated use of one-to-one correspondences.

CONCLUSION

The arguments presented here have put into question Piaget and Szeminska's interpretation of multiplication as being constructed on the basis of "multiple correspondence". Instead, an alternate model, using the iteration of a "one-to-many" relation as a basis for generating multiplicative situations, has been suggested. Various models involving intensive and extensive quantities, or the notion of ratio, are discussed in a companion paper.

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MODELS OF MULTIPLICATION BASED ON THE CONCEPT OF RATIO

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Multiplication being one of the most fundamental schemas in arithmetic, it is important to identify the simplest and easiest situations leading the child to its initial construction. In a companion paper (Herscovics et al, 1983), we investigated Piaget's suggested construction based on "multiple correspondence", that is a generalization to n sets of the composition of equivalences based on one-to-one correspondences. In that paper, we presented arguments which suggested that multiplicative situations could be constructed in a much simpler way by an iteration of a one-to-many relation. Two other important studies have dealt with the concept of multiplication, the first one distinguishing between intensive and extensive quantities (Schwartz,1976), the second one basing multiplication on the concept of ratio (Vergnaud,1983). The present communication examines whether or not these last two models could in fact represent the simplest and most primitive multiplicative situation.

SCHWARTZ'S ANALYSIS

In his study of the semantic aspects of quantity, Schwartz distinguishes between the use of number as a noun (e.g. the sum of two and three is five) and the use of number as an adjective (e.g. two apples and three apples are five apples). Even if it may have some interesting implications for arithmetical operations, one must nevertheless examine the merits of such a distinction. Indeed, can we really perceive number as merely a noun, that is, without referent? Even for the adult, is the word "three" simply a noun? In fact, when we use it without tying it explicitly to a concrete referent, we implicitly identify it with the class of all possible triplets. Hence the referent is an equivalence class which is independent of the nature of the objects, a class which is the outcome of a long process of generalization of the referent. This construction of number shows clearly that it cannot exist without referent but that on the contrary, it results from a process of abstraction through which we detach ourselves from any example in particular.

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Going back to one of Lebesgue's ideas, Schwartz also distinguishes between "intensive attributes" (chemical composition, density, temperature, etc.) and "extensive attributes" (volume, surface, total mass, etc.) of continuous quantities and notes that "extensive quantities" can be added whereas "intensive quantities" cannot. However, another feature distinguishing the two types of quantities is whether their measure is global or local: assuming a homogeneous composition, the measure of an intensive quantity can be obtained from any part whatsoever of the object, whereas in the case of an extensive quantity, the whole object in its entirety must be considered. Without any doubt, such a differentiation is important in the case of continuous quantities, but the question must be raised as to whether or not it applies to the quantification of discrete objects.

In his analysis of the multiplicative situation constituted by the following set of twelve points, Schwartz perceives the factors $\{4, \text{points/row}\}$ and $\{3, \text{rows}\}$ as intensive and extensive quantities respectively. However, the interpretation of points/row as an intensive quantity does not satisfy the criterion suggested above, whereby any portion of the object in question could be used to measure it. Indeed, a row or column do not constitute arbitrary portions of the set. And to select them as factors presumes a prior awareness of the presence of equivalent groupings. Thus, the notion of grouping necessarily precedes that of intensive quantity within the context of discrete objects.

x x x x
x x x x
x x x x

Schwartz argues that the expression $3 \times \{4, \text{points}\} = \{12, \text{points}\}$ cannot be used to represent the preceding twelve points since, according to him, the numeral 3 in this expression is but a noun without any referent. This view seems particularly mistaken since, in the present context, the numeral 3 represents the number of rows. In his attempt to apply a model based on continuous quantities to discrete ones, Schwartz's interpretation of multiplication has proved to be deficient. Nevertheless, this analysis has brought out the prevalence and precedence of the notion of equivalent groupings to that of intensive quantity, in the context of discrete objects.

Using Schwartz's distinction between intensive and extensive quantities, Quintero (1981) has investigated the semantic understanding of word problems involving multiplication problems of the intensive x extensive type. Her results and conclusions are in line with those of other researchers studying problem solving to the effect that the modeling of word problems constitutes a major cognitive obstacle. Even if she did not study the multiplication operation per se, her choice of word problems is most interesting. Although she considers 12 doll dresses in each box, 22 hits per baseball game, 22 children per classroom, to be "intensive quantities", within the context of multiplication these quantities can equally well be considered as "equivalent sets". But these examples bring out the varied contexts (spatial, temporal, social) in which these groupings occur. And it would be most interesting to study how the different contexts affect the notion of equivalent groupings for the young child.

Quintero also includes problems such as "22 candies per quarter" and "22 lollypops for a dollar". She considers these to be intensive quantities but of a more abstract level. However, the nature of these problems seems to be somewhat different from the previous ones since they involve the concept of ratio. This concept of ratio has been the basis of Vergnaud's classification of various multiplicative situations.

VERGNAUD'S ANALYSIS

Vergnaud perceives multiplicative structures and situations as a conceptual field involving many interconnected concepts (fraction, ratio, linear function, vector space, etc.) whose acquisition takes many years. While his studies have dealt with students aged 11 to 15, the question arises as to whether or not his models based on the notion of ratio, are applicable to a simple multiplicative situation involving the quantification of a set of equivalent groupings.

M_1	M_2
1	a
b	x

Let us examine his most elementary model of multiplication viewed as an isomorphism of measures, that is as a direct proportion between two measure spaces M_1 and M_2 illustrated by the example "Richard buys 4 cakes, 15 cents each. How much does he have to pay?" where $a = 15$, $b = 4$, $M_1 = \{\text{number of cakes}\}$, $M_2 = \{\text{costs}\}$.

According to Vergnaud, there are four ways of solving such problems. The first one, which he calls a "binary law of composition" ($a \times b = x$, where a and b are perceived as numbers, and not as quantities), the next two, "unary operations", involve either a "scalar operator" ($a \xrightarrow{\times b} x$) or a "functional operator" $(b \xrightarrow{xa} x)$.

$$\begin{tikzcd} M_1 & M_2 \\ 1 \arrow[d, bend right, "xb"'] & a \arrow[d, bend left, "xb"] \\ b & x \end{tikzcd} \qquad \begin{tikzcd} M_1 & M_2 \\ 1 \arrow[r, "xa"'] & a \\ b \arrow[r, "xa"'] & x \end{tikzcd}$$

The fourth way would be an iteration of addition, a + a + a +....(b terms) which he does not consider to be a multiplicative procedure.

Of course, this iteration of addition can be interpreted as a perception of a multiplicative situation as soon as it is viewed as "b times a". On the other hand, his binary law of composition is subject to the same criticism as Schwartz's "number as a noun". For indeed, in the case of natural numbers a and b, a x b implies a number a of sets of b elements each (or b sets of a elements). And in this context, the validity of Vergnaud's model based on ratio is questionable.

Does the child who perceives a multiplicative situation when generating "four packages of three cards each" resort to the concept of ratio? Not necessarily, since he can use a much more elementary notion, that of equivalent sets. One can argue that the notion of ratio is present implicitly since the construction of such sets calls on the one-to-many or the many-to-one relation (3 cards in each package). But does this relation coincide with the concept of ratio? Ratio is a comparison of two quantities which is expressed as the quotient of their two measures (e.g. 12 candies for 3 children yields a ratio of 12:3; John's 5 flowers compared with Paul's 7 flowers yields a ratio of 5:7). The one-to-many or many-to-one relation can be considered a particular case of ratio, in which one of the terms is 1, only when comparison is intended and two measure spaces are present.

One can of course claim that there is always the ratio between a set and the number of its elements. In fact, one can also, in the extreme, see a ratio in an even simpler task such as "give me three cards" if it is perceived as a set of three cards. But why does one feel uncomfortable to accept this as a ratio? It is precisely because in such situations no comparison is intended, and we do not have two distinct measure spaces. In fact, in such a situation, the package of three cards is but another unit of measure in the quantification of the set of cards. There is here only one measure space, that of the number of cards, and these are measured either in simple units (a card), or in larger units (a package).

THE QUANTIFICATION OF EQUIVALENT SETS

The generation of a multiplicative situation based on equivalent sets of identical objects does not necessitate the use of ratio. The construction of equivalent sets involves the one-to-many relation as for instance in the task "Can you make three rows of seven cards" which requires the use of this relation in setting up the three rows. The child can use three main procedures: 1) count out three rows of seven; 2) count out the first row and then proceed by one-to-one correspondence for the other two; 3) count out the first row and then proceed to complete the columns using a one-to-two correspondence. Of course, such a situation tends towards becoming a multiplicative one only when it is perceived as three times seven. And with such an objective, the three procedures are not equivalent since it is only the first one which explicitly preserves and iterates the "row of seven", while in the other two procedures the "row of seven" could possibly be lost from sight when setting up the various correspondences.

Quite cautiously, we used the expression "tends to become a multiplicative" situation since even if the child expresses it as "three times seven" he may but verbalize the number of times he has gathered seven cards. And we would not consider this a perception of a multiplicative situation. The word "times" is used very early by the child in the process of counting actions as for instance in skipping rope. And the counting of the number of equivalent sets may only represent a counting of actions. It is only when the focus shifts from actions to the equivalent sets produced by such actions and that these equivalent sets are perceived as parts of a whole that we can truly refer to a multiplicative situation. And this is what we would consider an intuitive understanding of multiplication.

Of course we do not know how this shift from actions to equivalent sets occurs. We don't even know how the child constructs equivalent sets. As mentioned earlier, this construction involves the one-to-many relation. But clearly there is more to it than that. Equivalent sets also involve one-to-one correspondences just as they involve a minimal concept of number when counting the number of groupings in a multiplicative situation. But are conservation of number and conservation of one-to-one correspondence (in the Piagetian sense) essential for an initial construction of multiplication? Piaget's model based on "multiple correspondence (see companion paper) is much too sophisticated to be taken as an initial construction.

CONCLUSION

In our attempt to describe the child's construction of multiplication, we asked ourselves what would constitute an intuitive level of understanding of this conceptual schema. An examination of Piaget's pioneer work has made us aware that such a construction is not independant of the multiplicative situation presented to the subject. In Piaget's case he provided the child with a situation whose multiplicative nature could only be extracted through "multiple correspondence". We thus tried to identify the most primitive multiplicative situation likely to lead the child to the simplest possible construction. The traditional quantification of equivalent sets of discrete objects seemed to provide the most primitive situation. Nevertheless, in order to assert this, it had to be confronted with the other two existing models, Schwartz's model based on intensive/extensive quantities and Vergnaud's model based on ratio.

Although we have concluded that the quantification of equivalent sets is indeed the most primitive situation, it has proved to be far from a simple one. And we may have raised more questions than found answers. In the last four months we have been working with Carolyn Kieran and Bernadette Ska on exploratory studies investigating some of these problems. But we have found it most difficult to design the appropriate tasks and to find the questions which would reveal the child's thinking without feeding him the sought for answers. However such difficulties are unavoidable when studying intuitive understanding for it cannot be uncovered by a mere analysis of arithmetical skills.

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DIAGNOSTIC TEACHING OF ADDITIVE AND MULTIPLICATIVE PROBLEMS

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This paper forms an introduction to the two following. The three papers link with last year's paper to PME (Bell, 1982) and describe further experiments in designing teaching to deal with diagnosed misconceptions. Two experiments will be referred to in some depth (by my two colleagues), and some aspects of two others will be commented upon here.

Our main field of interest is that of the learning of additive and multiplicative structures and their application to everyday situations. In previous years we have studied additive relationships with directed numbers, and decimal numbers with particular emphasis on multiplicative relationships. This defines our field from the mathematical standpoint. From the psychological point of view, we are studying the teaching design variables of structure, context, feedback and cognitive conflict.

ADDITIVE PROBLEMS - TEACHING EXPERIMENT

The experiment on Additive Problems focussed on the ability to identify and perform the correct operation (addition or subtraction) in verbally presented problems in situations such as usual price, discount and sale price, starting and finishing times (of a TV programme) and length, and a number of other everyday situations.

We were interested in the relationship between ability to identify the operation with the numbers given in the problem, and the awareness of the general relationship among the quantities, as expressed in a verbal generalisation. An example of one of the test questions is given.

Programme Times

1 Julie's TV programme starts at 8.15 and finishes at 8.50.

How long is it? ☐

2 Her brother's programme is 25 minutes long and finishes at 9.45.

When does it start? ☐

3. How do you work out the starting time from the length and the finishing time? Fill the gaps.

(from/to)

You ________ the __________ __________ the __________.

4. Fill the boxes with +, -, =, to make different true statements; when no more are possible, write N in the boxes.

FINISH TIME ☐ LENGTH ☐ START TIME

FINISH TIME ☐ LENGTH ☐ START TIME

START TIME ☐ FINISH TIME ☐ LENGTH

START TIME ☐ FINISH TIME ☐ LENGTH

* 45 mins and 9.55 in post test

The mean success rate on this and a similar question was approximately 50% for the numerical question (no 2), 20% for the word formulae and 10% for the verbal statement. (The class was a low-attaining group of 13 year olds.)

Some of the difficulties concerned the order of subtraction; others appeared to derive from the need to reverse the direction of thought to recover the starting time from the other data. The hypothesis on which the teaching was based was that improvement would result from awareness of the three forms of relation existing in this and similar problems - S + L = F (Start time + length = finish time), F - S = L and F - L = S; with identification of which quantity (F) was the sum of the other two. This hypothesis amounts to teaching the key concepts and generalisations which relate to the task. Intrinsic feedback was built into the learning tasks, by relating each numerical problem to its corresponding formula, and checking that the three correct formulae were the ones being used; conflict was to lead to correction.

The teaching covered a number of contexts, in each of which the three forms of the add/subtract formulae were emphasised.

Several interesting observations were made. Often in the attempt to give a verbal statement, pupils had to use specific rather than general labels; they could not say 'I'd take the length from the finishing time' but said, for example, 'I'd take, like, the 25 from the 45'. The programme times questions were much harder than similar ones concerning usual price, sale price and discount. In interviews several pupils showed much difficulty in visualising or reasoning with clock faces. But it may be that the important difference is that the programme times situation contains two kinds of quantity, the times, which are position markers, and the duration, which is an extensive quantity.

The results showed improvements in the production and identification of the verbal formulae and generalisations, but not in the numerical questions. The checks were partly successful; some pupils failed to appreciate the need for independent derivation of the two answers if the check was to be valid. A full account of this experiment is given in Bell and Low (1983).

TRANSFER EFFECTS IN THE TEACHING OF DECIMALS

Of the two experiments performed in this mathematical field, the first concerned the teaching of the four operations, with attention to multiplication and division by small numbers and the estimation of results. This was an experiment with a single middle-ability class aged about 14, where interest focussed on the relation between the teaching and the test gains, in particular on the extent to which 'transfer' gains took place on items not directly covered in the teaching.

In this experiment, the main findings were

1 that fairly intensive teaching, focussed sharply on known misconceptions was successful in improving performance, but
2 there was very little transfer to points not so strongly focussed upon, even though they were dependent on the same general concepts of place value.

For example, the error of counting scale spaces without regard to whether they represent tenths, so labelling 5.4 on a scale of fifths as 5.2, received explicit attention and was substantially improved, but that of omitting zero, labelling 2.03 as 2.3, was hardly affected. Similarly, the error of giving 6.2 x 10 as 6.20 was not shifted at all, in spite of a strong emphasis on estimating sizes of answers, which made a very substantial improvement to items such as 19.5 x 5.4 = .1053 / 1.053 / 10.53 / 105.3 / 1053; these were superficially quite similar to the tasks used in the teaching. It is of interest to consider further experiments based on some of these items to study whether changes in the teaching can produce greater transfer. One might try greater linking of the necessary specific teaching on the misconceptions with the general idea of place value, and a more general emphasis on checking the sizes of results against expectations in all situations.

The second decimal teaching experiment which is reported more fully in Swan's paper below, did, in fact, produce somewhat greater transfer effects under one of the two teaching methods which it compared. (These were a conflict-discussion teaching method and a 'positive-only' approach. The latter method was 'diagnostic' in that it focussed on the known misconceptions, but it taught procedures which avoided rather than exposed them).

The transfer effects observed were to items asking for the amounts shaded on squared paper diagrams on which 100 small squares made one large unit square. Improvement in the ability to recognise, for example, a diagram representing 1.08 units and resist the distraction of counting small squares and giving 1.8, was considerably greater in the conflict than in the positive-only group. In both groups the teaching had been confined to the number line, area not being considered, so this is a transfer, from line to area, of success in overcoming the 'zero as a placeholder' misconception.

The main result of this experiment, however, was of considerable gains by both groups, with a significantly greater gain in the 'conflict' group. In both cases, the gains were closely related to the specific content of the teaching; transfer across even minor perceptual or structural differences was not observable. The differential transfer commented upon above was the exception.

A COMPARISON OF CONFLICT-DISCUSSION AND POSITIVE-ONLY DIAGNOSTIC TEACHING METHODS

The transfer effects in this experiment have been discussed above. However, this was not its main aim, which was to make a direct comparison of two variants of our diagnostic teaching. Both of these were diagnostic, in that the lessons each focussed on one known misconception, such as treating the decimal point as a separator of two integers, counting marks in scale reading, ignoring whether they were tenths, and so on. In the positive only method, correct procedures were immediately taught for avoiding the errors, while in the conflict discussion method, the pupils were first placed in a situation which gave rise to the misconception, thus provoking a conflict which was then discussed. This experiment gave positive evidence of the value of the conflict approach to diagnostic teaching.

DETERMINANTS OF RELATIVE DIFFICULTY IN CHOICE OF OPERATION PROBLEMS

The experiment to be reported by Greer is a more intensive study of multiplicative problems with decimals, involving choice of operation. These were target problems in some of the research of the previous project, but it proved that difficulties in understanding decimal notation and operations had to be investigated, and teaching materials designed, before methods for choice of operation could be given full attention. The present experiment is a study of pupils' difficulties with choice of operation, and was a preliminary to the design of teaching material.

In this experiment, 12 and 13-year-olds were tested with two types of task:

(i) writing down calculations required to solve verbal problems
(ii) making up stories to fit given calculations, to test their understanding of multiplication and division of positive numbers.

Selected pupils were interviewed to investigate further the thinking processes involved. The results indicate

(a) the pervasive nature of certain numerical misconceptions
(b) the effects of structual differences among the items; particularly whether multiplication can be conceived as repeated addition or not, and whether division has the structure of partition, quotition or rate, and

(c) specific effects of context, attributable to such aspects as relative familiarity, as well as a number of interactions between these three sets of factors. These studies are reported in the paper by Greer.

TEACHING PACKAGES

The material used in the Decimal teaching experiments has been collected into a Teachers' Handbook. This contains extensive teachers' notes, pupils' worksheets for duplication, the diagnostic test and a markscheme designed for extracting information regarding the misconceptions held by the pupils. A video tape showing pupils who do and do not hold the misconceptions accompanies the pack.

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TEACHING DECIMAL PLACE VALUE

A COMPARATIVE STUDY OF "CONFLICT" AND "POSITIVE ONLY" APPROACHES

Malcolm Swan, Shell Centre for Mathematical Education

Recent large scale surveys (see Hart (1981), APU (1980, 1981) and our own research, for example, Bell et al (1981)) have revealed that many children possess fundamental misconceptions concerning the nature and use of decimal numbers, and that these misconceptions appear to remain largely unaffected by traditional arithmetic-based courses. At the Shell Centre, we have been trying to produce and evaluate principles for the design and use of teaching materials which enable children to overcome such misconceptions. This paper will describe one experiment which illustrates the value of a diagnostic teaching approach which focusses sharply on known misconceptions and makes these explicit to the children. (It has been more fully written up in Swan (1983).) Two teaching styles were examined. The first of these, the conflict teaching approach, was intended to involve the pupils in discussion and reflection of their own misconceptions and errors, thus creating an awareness that new modified concepts and methods were needed. There was therefore, a 'destructive' phase, in which old ideas were shown to be insufficient and inaccurate, before new concepts and methods were introduced. The second teaching style, the positive only approach, made no attempt to examine errors, and in fact avoided them wherever possible by teaching the pupils to use simple and efficient methods from the start. These methods were then practised intensively. So far as was possible within the constraints imposed by this dichotomy, the same teaching material was used with both groups. The relative effectiveness of the two teaching methods was monitored by a pre-test, an immediate post-test and a delayed post-test (3 months later) together with observations made during lessons.

IDENTIFYING THE COMMON MISCONCEPTIONS AND ERRORS

The following (abbreviated) list of common misconceptions was evolved from previous research (our own and that of the CSMS and the APU), and formed the basis for our diagnostic test.

1 Verbalising a decimal Many children read decimals incorrectly. Some appear to ignore the decimal point altogether, while others read the number as if it were composed of two independent integers separated by a mere 'dot' (Thus 12.62 is read as 'twelve point sixty two'). Such children are often unable to relate the digits before and after the

6

decimal point and produce such answers as 0.8 + 0.2 = 0.10, 3.1 x 10 = 30.10 and 0.3 ÷ 2 = 0.1½. Others reveal that they do not recognise the denary nature of decimal numbers when they confuse the decimal point with other 'separators' such as the 'r' in 9 r 2 (nine remainder 2) or the fraction bar in 3/5. (It is also important to realise that children may be merely mimicking undestanding when they recite the correct fractional place headings.)

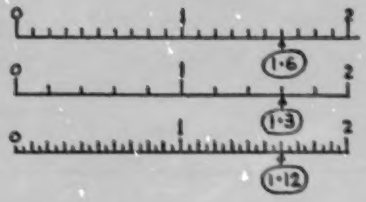
2 Comparing Decimals In one pilot study, we invited 98 mixed ability 12 and 13 year old pupils to select the largest of the three numbers 0.62, 0.236 and 0.4. 17% chose 0.62, 50% chose 0.236, and a surprising 28% chose 0.4 (because "it only goes back as far as tenths, while the others go back as far as hundredths and thousandths" or, more rarely, because $\frac{1}{4} > \frac{1}{62}$ and $\frac{1}{236}$). Many children appeared to feel that they could correctly compare decimals by simply examining their 'lengths'. Thus either 'longer' (more digits) or 'shorter' numbers were always greater in value.

3 Using Zero as a Place Holder Only 51% of 15 year old pupils could correctly complete the following APU item:

73.45 = 70 + 3 + 0.4 + ☐

The most common errors, 5 (16%), 0.5 (8%), and 0.41 (5%) reveal that the misconceptions stated above were at least partly to blame.

4 Scale Reading As well as being an important skill in its own right, scale reading discriminates very clearly between pupils who have a deep, genuine understanding of decimals and those who do not. Frequently, in our interviews, pupils ignored the number or value of the intervals between marked calibrations, and produced such contradictory answers as these:



5 The "Denseness" of Decimals The CSMS research revealed the difficulty that children have in producing a decimal which lies within a given interval, and in appreciating that there exist an infinite number of possible choices. (Only between 12% (12 year olds) and 20% (15 year olds) recognised that there exist 'lots', 'hundreds' or an 'infinite' number of decimals between 0.41 and 0.42.)

6 Application of Decimals The CSMS research invited children to embody decimals in stories which could accompany given calculations (eg, 6.4 + 2.3 = 8.7). It was interesting to note that only between 33% (12 year olds) and 41% (15 year olds) could produce satisfactory stories, and many appeared to be influenced by artificial stereotyped text-book questions, which had little to do with the real world. These stories often included quantities (eg, sweets) which are not, in practice, subdivided.

THE ORGANISATION OF THE TEACHING EXPERIMENT

Two parallel classes of second year (12 and 13 year olds) pupils were chosen from a suburban comprehensive school in Nottingham. These classes were from the upper ability band of one half of a year group, and were chosen so that their performances on the pre-test were reasonably comparable, both contained a wide spread of ability, and both included just a few pupils who were competent in the area of place value with decimals.The same teacher (the author) taught both classes for eight one-hour lessons using the two teaching approaches. The content of the lessons in both groups can be roughly divided into three areas:

1) Completing Sequences This section attempted to provide the pupils with a 'concrete' model for decimal place value (the number line), and encourage the children to visualise this line when performing simple additions and subtractions. The correct verbalisation of decimals was emphasised.
2) Reading Scales These lessons attempted to enable the pupils to read a scale which had been subdivided into tenths, fifths or twentieths, and to interpolate successfully between marked calibrations.
3) Comparing Decimals Number lines were provided to enable pupils to correctly compare decimal numbers containing different numbers of digits. The pupils were then encouraged to formulate their own rules for comparing decimals without using the number lines.

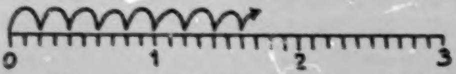
As this list makes clear, the teaching styles in both groups were essentially 'diagnostic' in that they focussed on the known areas of difficulty discovered by the pre-test, but only the 'conflict' group were asked to examine and reflect on the misconceptions and errors exposed by that test. Let us illustrate how this was done using the lessons on "completing sequences".

The 'Conflict' Lessons

Each lesson sequence involved 4 phases:

C1 **The 'Intuitive' phase** The pupils were asked to perform a task intuitively, either orally or by means of a worksheet, in order to expose their misconceptions (which had been revealed by the pre-test). *No attempt was made to correct any errors at this stage.* For example, they were given a worksheet containing ten sequences to be completed, such as 0.2, 0.4, 0.6, ..., ..., ... Many produced answers like 0.8, 0.10, 0.12 (verbalising them as 'nought point ten' etc).

C2 **The 'conflict' phase** Pupils were then given the same tasks *again*, but this time were also equipped with an easy alternative and understandable method, suggested by the teacher. This usually involved using a number line, together with a calculator check. In our example, the pupils were shown how the 'sequences' situation may be viewed as 'bouncing' along a number line:



They then repeated many of their earlier questions, and were encouraged to reflect on, debate, write down, and resolve the inconsistences revealed by their two sets of answers.

C3 **The 'Resolution' Phase** A discussion was then held to make children aware of the errors and misconceptions exposed by the previus phase. In one discussion, pupils stated that:

"There is no such number as nought point ten"

"Nought point ten can be exchanged for a whole number"

"Nought point one nought is the same as nought point ten"

(This latter comment was made by a pupil who had been exploring the sequence 0.05, 0.10, 0.15, ... with his calculator).

C4 **The 'Reinforcement' Phase** Correct concepts and methods were then reinforced and practised using carefully constructed exercises. Near the end of each section of work, pupils were asked to adopt the role of a teacher and mark an exercise which was full of mistakes (made by a fictitious 'pupil') and diagnose errors themselves.

The 'Positive Only' Lessons

Each lesson sequence only consisted of two phases:

P1 **The 'Teaching a Method' phase** The class were taught methods for producing correct answers, based on understanding. (They were the same methods as those adopted in phase C2 with the 'conflict' group.)

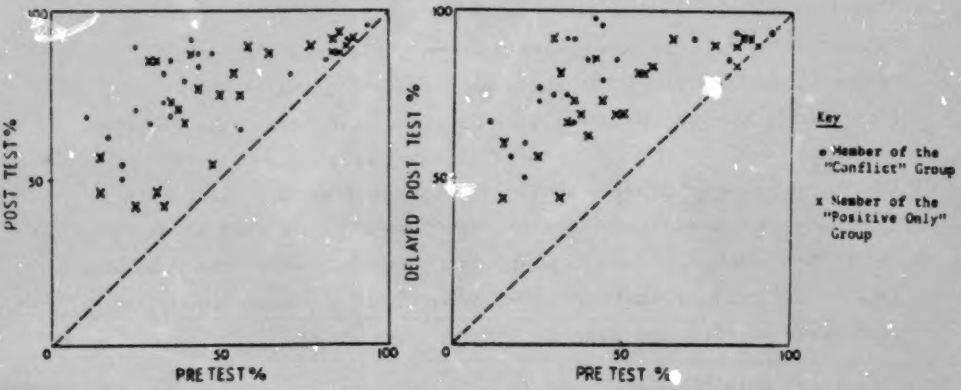
P2 **The 'Reinforcement' phase** These methods were then reinforced and practiced using similar exercises as those adopted in C4, except that pupils were never asked to mark work or diagnose errors. This group covered the work much faster than their 'conflict' counterparts, and so supplementary material was also provided.

RESULTS AND CONCLUSIONS

The overall performance of the two groups on the 48 test items is given below:

	Conflict Group (n = 22)		Positive Only Group (n = 25)	
	Mean	sd	Mean	sd
Pre-test	44.3%	11.7	52.0%	11.52
Post-test	77.7%	6.2	74.5%	8.33
Delayed post-test	80.2%	6.54	76.2%	7.08

It can be seen that both groups made substantial gains during the teaching, and these gains were retained until the delayed post-test. The overall gain by the 'conflict' group, 35.9%, was much greater than that made by the 'positive only' group, 24.2%, but before these gains can be reliably compared, account must be taken of the initial superiority of the 'positive only' group. The multiple regression program PMMD*SMLR (Youngman 1976) was used for this purpose. A correlational analysis (program PMMD*CATT, Youngman 1976) was then applied to these residual scores to detect differences. The results showed that the pre - post-test results were significantly different at the 10% level and the pre - delayed results were significantly different at the 5% level. The scattergraphs below illustrate the results:



These results enable us to draw the following conclusions:

Both the 'conflict' and 'positive only' teaching styles were very effective at enabling children to understand decimal place value, given that the teaching material was essentially diagnostic in both cases. The 'conflict' style was significantly more effective at 'permanently' removing and correcting most of the misconceptions that we described. The 'conflict' style did not appear to cause pupils who were already competent to become confused and regress when they were introduced to misconceptions that they themselves did not possess.

Additional, more tentative hypotheses also emerged from a closer analysis of individual test items:

A 'conflict' approach may lead to a deeper conceptual understanding. By discussing what a decimal is not, for example, our pupils seemed to have come to a deeper realisation of what a decimal is.

'Positive only' teaching is perhaps more likely to result in mechanical, rule based learning. For example, some 'positive only' pupils regressed on the item where they were asked to compare 5436, 547 and 56, and stated that 56 was the largest. Presumably, they were using a rule for comparing decimals digit by digit from left to right, as they would when comparing 0.5436, 0.547 and 0.56.

Both teaching approaches may lead to some learning which transfers to an untaught context. (One question contained such a context, where the decimal numbers were embodied as areas. The 'conflict' group achieved a greater improvement on the item which involved using zero as a place holder.)

Conflict lessons may make greater managerial demands on the teacher, not least because of the considerable debate (and noise) that is created when pupils are actively encouraged to make and verbalise errors.

We conclude by emphasising that although 'conflict' teaching is difficult to design, and time consuming to perform, it appears to have a considerable pay-off in terms of a deeper understnading and a greater awareness of errors to avoid.

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D. COGNITIVE STUDIES IN GEOMETRY

1. *CONCEPT FORMATION IN GEOMETRY*
2. *SPATIAL VISUALIZATION*
3. *REASONING*

SOME PERCEPTUAL INFLUENCES IN LEARNING GEOMETRY

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INTRODUCTION

Learning geometry usually begins with the perception of geometric figures. The visual system is very adept at recognizing geometric figures and distinguishing details of geometric importance. However, what one sees can sometimes distort a mathematical concept.

Often some features of a figure are more visually compelling than others. For example, the visual impact of each of the squares in Figure 1 is quite different.

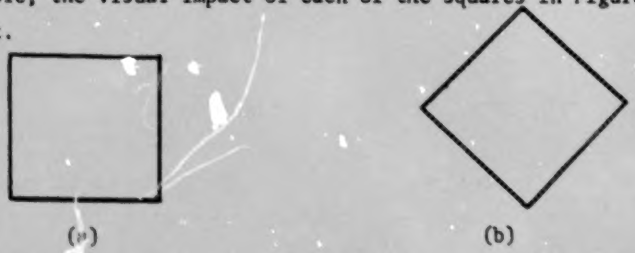

(a) (b)

Figure 1

In Figure 1a, the right angles, equal sides and parallel opposite sides are clear, but symmetry about the implicit diagonals is not obvious. On the other hand, in Figure 1b, symmetry about the implicit diagonals is clear, but the features of right angles, equal sides and parallel opposite sides is not prominent. "Thus, there is a trade-off; any particular representation makes certain information explicit at the expense of information that is pushed into the background and may be quite hard to recover. This is important, because how information is represented can greatly affect how easy it is to do different things with it," (Marr, David 1982).

Orientation of figures and/or vertical symmetry of figures are important visual features for identifying, classifying and grouping figures in perceptual tasks (for example, Brown, Hitchcock and Michels 1962, Goldmeier 1937, Takala 1941, Zusne and Michels 1962). Some work suggests that in coding figures subjects conceive an upright prototype together with information about recognizing the prototype in a tilted position (for example, Attneave 1968, Steinfeld 1970). Other times the upright orientation of the conceived prototype may be so important to the subject that he has difficulty working with the prototype figure in another orientation, (Braine 1973). Similar difficulties can arise in doing mathematical work. Some students have difficulties recognizing perpendicular lines embedded within a complex figure if the lines are not in the vertical-horizontal position, even though the students are told that the lines are perpendicular, (Zykova 1969).

The perceptual role of orientation in recognizing figures suggests several questions about the influence of figure orientation in learning a geometric concept. For example, is it best to present upright figures first since they are more easily recognized? Is a bias for upright figures due to insufficient examples of tilted figures? Can students deduce upright prototypes if they are shown only tilted figures for a concept?

THE STUDY, RESULTS AND OBSERVATIONS

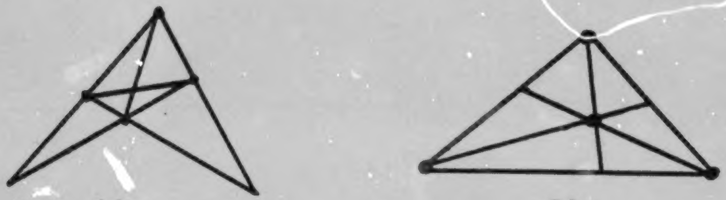
The present study was concerned with some of the questions of how the orientation of illustrative figures in teaching a geometric concept influence the learning of the concept. Three different concepts were taught: an altitude of a triangle from a vertex to the opposite side, an angle of incidence from a point to a line and its corresponding angle of reflection, and a complete 4-point and its three diagonal points. Four types of instructional booklets were used for each concept. Type a booklets contained all upright figures, type b booklets contained all tilted figures, type c booklets contained upright figures in the first half and tilted figures in the second half, and type d booklets contained tilted figures in the first half and upright figures in the second half. All the booklets for one concept contained the same written material and the same sixteen figures in the same order. Each student was instructed in only one concept. Immediately after studying the instructional booklets, students were given test booklets. Students had to identify correct illustrations and complete figures of the concept. Half of the test figures were upright and half were tilted. The students were 36 [illegible] graders, 65 9-th graders and 67 college students in a pre-Calculus, Algebra and Trigonometry class.

Combined scores for the 9-th graders and college students were analyzed using a grade X concept x instructional format repeated measures ANOVA. The

The analysis showed no significant differences among the instructional formats. However, students scored significantly better on upright test figures than on tilted test figures. Further, not only were the three concepts significantly different to learn but also the interaction of figure orientation and the concepts was significantly different.

For the combined group of 9-th graders and college students the highest mean was the altitude of a triangle, the next highest mean was the angle of incidence and the lowest mean was the complete 4-point. The bias in favor of upright figures was strongest for the altitude of a triangle concept and weakest for the complete 4-point concept. Thus the bias in favor of upright figures was greatest for the concept that was learned best. For the 6-th graders, the highest mean was the angle of incidence, the next highest mean was the complete 4-point and the lowest mean was the altitude of a triangle. The means for the complete 4-point concept of the 6-th graders and of the combined 9-th graders and college students were very close.

It is worthwhile to note learning patterns that were observed from studying the individual test booklets. For each concept there was a type of figure that was particularly difficult to learn. For the altitude of a triangle concept, exterior altitudes from acute vertices in an obtuse triange were difficult for some students. For the angle of incidence concept, a 90° angle of incidence, where the angle of incidence and the angle of reflection have a common side, was difficult for some students. For the complete 4-point concept, some students had difficulty incorporating the two general types of figures into the concept. Some students had difficulty learning the three pointed star figures, as shown in Figure 2a, whereas others had difficulty learning the triangular figures, as shown in Figure 2b.



(a) (b)

Figure 2

DISCUSSION

The perceptual model of coding upright prototype figures is relevant to learning geometric concepts. It is suggested that it can be beneficial for a student to have a good upright prototype of a class of figures and some strategy for recognizing the prototype in a tilted configuration. If figures have good visual clues for uprightness or vertical symmetry, the figures may be coded more easily than figures with weak upright clues. Thus a concept may be difficult because illustrative figures present visual coding problems. For example, the complete 4-point concept is the simplest geometric concept of the three in the study, (it uses only the ideas of point, straight line and incidence of points and lines), but it was the most difficult concept for the 9-th graders and college students.

In addition, sometimes a visual property is so compelling to the viewer that the property is maintained even when it contradicts the stated mathematical concept. The erroneous construction of "altitudes" which were not perpendicular to a side of a triangle but were in the interior of the triangle is an example of this problem.

It is also difficult to bridge visual categories. For example, the star figures and the triangular figures for the complete 4-point concept are visually different types of figures. The power of the geometric concept is that it creates conceptual unity where there is visual diversity.

Perception is a powerful tool in learning geometry, but it also creates prpblems when visual learning conflicts with mathematical learning. This conflict can be a real obstacle for students and it deserves serious considerations from educators.

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THE ROLE OF CRITICAL AND NON-CRITICAL ATTRIBUTES IN THE CONCEPT IMAGE OF GEOMETRICAL CONCEPTS

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Shlomo Vinner - The Hebrew University Jerusalem

INTRODUCTION

This study is a continuation of our research on the formation of geometrical concepts (Vinner and Hershkowitz 1980, Hershkowitz and Vinner 1982, and Vinner and Hershkowitz 1983). We used the term *concept image*(i.e. the concept as it i reflected in the student's mind), which may be different from the concept itsel (i.e. the concept as it follows from its mathematical definition).

We found that:

1. The *concept image* has a definite structure (the C.C.P.) which is common to t whole of the population we examined. This is true for concepts which have been part of the student's experience for many years, as well as for concept newly formed. At least one important question was left for further research does this C.C.P. describe the development of the concept image with age?
2. The verbal definition followed by some related activity, is a powerful tool in the concept formation of new concepts. On the other hand, for familiar concepts the verbal definition has only a small effect on the performance of certain tasks concerning certain "types" of concepts.

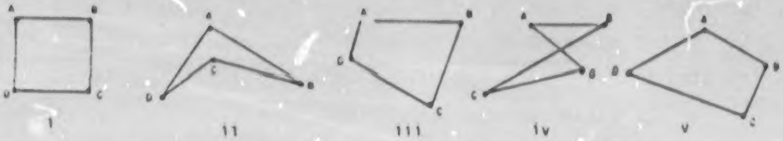
In this study we examined a larger population using a questionnaire similar to the one in previous studies. The subsequent analysis took into account such variables as age (grades 5, 6, 7 and 8), home background (culturally different students and others), ability levels in mathematics and sex. Here we will brin a few examples of the analysis with respect to age. By these examples we wish to add a somewhat deeper dimension to these studies, and this by using *concept analysis* (Herron 1983). What we mean by *concept analysis* will be explained in the examples.

ANALYSIS OF RESULTS AND DISCUSSION

Example 1

The following item was given to the whole population.

Item I. *Among the following shapes indicate those which are quadrilaterals:*



For each shape that is not a quadrilateral explain why.

The concept examined here is *quadrilateral.*

CONCEPT ANALYSIS. *Critical attributes* of the concept are those attributes which "must be present in order for an instance to be an example" of the concept (Herron). Suppose we define a quadrilateral as *a closed four-sided figure,* then "closed" or "four sided", are critical attributes included in the given definition of a quadrilateral. Other critical attributes "four vertices", "four angles", "two diagonals", follow from the definition, although they are not included in it.

In this item we have 3 examples of the concept (shapes i, ii, and v). One shape (iii) is a non-example because it lacks the critical attribute of closure, for example. The remaining shape (iv) is either an example, if we choose to define quadrilateral as above, or a non-example if we choose the definition: *A closed four-sided figure whose sides do not intersect.* *(For the sake of mathematical exactness, we should state that our definitions are "school geometry" definitions. At a more advanced level, for example, when the quadrilateral is taken as the "complete quadrilateral" there are other critical attributes.)* *Non-critical attributes* are those "which vary across examples" of the concept. For instance: the square (shape i) has some attributes, e.g. "four equal sides" or "four right angles", which other quadrilaterals do not have. The percentage of students in the different grades who indicated an example as a quadrilateral is given in Figure 1.

There is clearly a great improvement in identifying shape v (convex quad.) and shape ii (concave quad.) with age, from grade 5 to grade 7, and then effectively no change from grade 7 to 8. (This behavior of no change or even slight

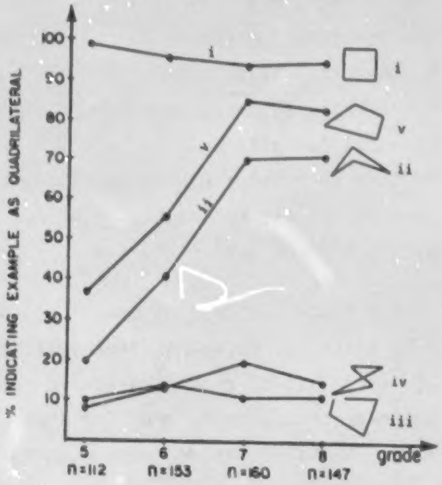


Figure 1

decline from grade 7 to 8 was found in the responses to most of the items, and demands some further research.) On the other hand, at all ages, most of the students identify a square as a quadrilateral. "Constant behavior" is also found in the identification of the "non-closed" shape (iii) and of shape iv, which has intersecting sides. For most of the students, in the different ages, these two shapes are non-examples. Does this mean that students in all the examined ages have the same cognitive or thought pattern concerning these two shapes? We can answer this question by analysing the different reasons which students gave when they decided that a given shape is not quadrilateral.

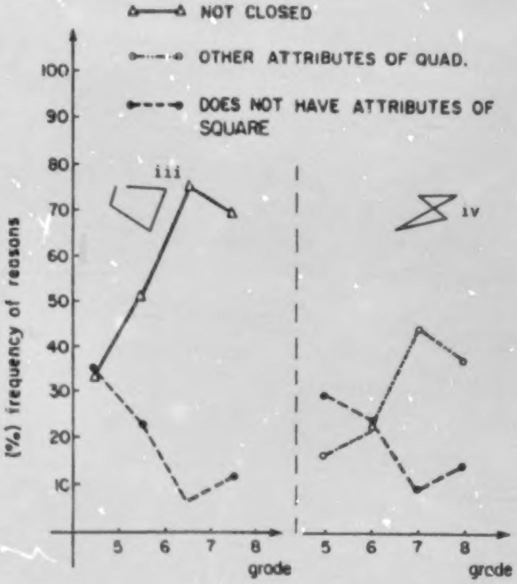


Figure 2

In figure 2 we see the change with age in the percentage frequency of the different reasons for the two shapes. There are two kinds of reason which have an opposite pattern of change. For shape iii we have a sharp increase in the frequency of reasons like "The shape is not closed and therefore is not quadrilateral" (which is correct), from grade 5 to grade 7 (from 34% to 76%), and a small decline from 7 grade to 8. On the other hand, we have wrong reasons, based on non-critical attributes - in this case, square attributes, like: "The four sides are not equal" or "The four angles are not right angles" or "The shape is not square", which decrease with age (from 36% in grade 5 to 7% in grade 7 and a small increase in grade 8).

For shape iv, reasons based on other attributes of the quadrilateral, like: "The shape has more than four vertices" or "more than four angles", increase with age (from 17% of reasons given in grade 5 to 45% in grade 7 and again a slight decrease from 7 to 8). As for shape iv the percentage of reasons based on the square's attributes decrease with age (except for grade 7 to 8). (It is worth noting that only 5% - 10% in each grade give reasons like: "It is not a quadrilateral because the two sides intersect," which is a critical attribute according to the second definition above, which sees this shape as non-example.)

We can sum up these results using Van Hiele theory. The number of students who judge an instance as an example or non-example of a concept, using critical attributes as criteria, increases with age. In other words the number of students who see the concept as the "bearers of their properties and recognise figures by their properties", (Wirzup 1976, the second Van Hiele's level), increases with age. On the other hand, the number of students who judge an instance as an example or non-example of a concept, by making a comparison with a prototype example existing in their concept image, decreases with age. In other words, the number of students who "judge figures according to their appearance" (Van Hiele's first level) decreases with age. If we look at the distribution of students on the Guttman Scale in each grade (figure 3), we see that

1) The distribution of students at different stages of the scale moves upwards from grade 5 to grade 7; that is, a decreasing number of students have a concept image consisting of the square only, and an increasing number have at least three shapes in their concept image of a quadrilateral.

2) For each grade level there is a highly structured C.C.P. - Coefficients of Reproducibility (C.R.) are very high.

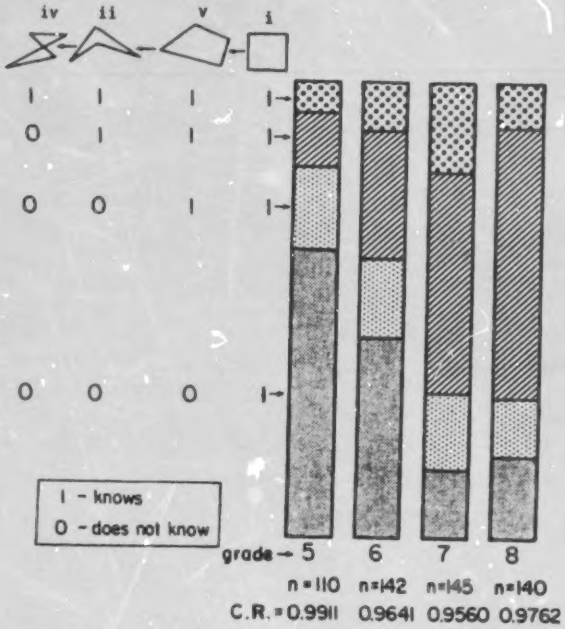


Figure 3

This example (and other examples with very similar results) show that the Common Cognitive Path (C.C.P.) found in previous studies, describes faithfully the development of concept image with age.

Example 2 - The following item was distributed in two versions.

The item in version 1 was

Definition: A polygon is a closed shape bounded by line segments. Indicate which of the following figures are polygons. (There were 15 figures of polygons and non-polygons.)

The item in version 2 was the same but without the polygon definition. The 15 different shapes can be classified into several categories: "regular" polygons (concave or convex), polygons which have a dominant non-critical attribute (e.g. triangle, rectangle), figures which are not closed, and figures which are closed but not bounded by line segments, e.g. the "cloud" (see figure 4) We found different response patterns for these categories. In figure 4 we bring only one example from each category, since the response pattern for other examples from the same category is similar.

It can be seen that a definition helps (if at all) only in the identification of figures which are not closed, and even then only the younger children.

In the case of closed non-polygonal shapes and shapes with dominant non-critical attributes, the definition actually seems to do "damage". For "regular" polygons the definition has no impact. Unfortunately, in this item, we did not ask the students to give reasons.

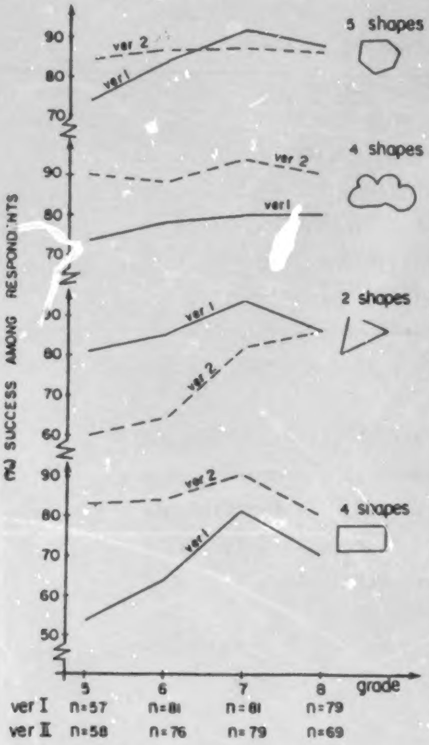


Figure 4

There are some concept examples, mainly very simple geometrical shapes, which students identify correctly at sight (first level), without considering critical attributes. This does not prevent the same students, for other less simple examples, making their identification using critical attributes (second level). The "regular polygon" is such a simple example, and therefore the definition, as a reminder of critical attributes, has no effect. Two critical attributes "closed" and "bounded by line-segments" appear in the definition. It appears that "closed" is more dominant and therefore helps in the identification of non-closed shapes, and disturbs in the "closed but not bounded by line segments" category. In the identification of the rectangle there is the disturbing effect of the non-critical attributes. Even students that naturally identify the rectangle (square, triangle etc.) as a polygon, in the presence of the definition (which, of course, makes no mention of non-critical attributes so evident in the given shape), they become confused and regard the non-critical attribute as infringing the definition.

CONCLUSION

The examples presented in this paper (the study contained other interesting examples) indicate that the student concept image matures with age and that different individuals have the same path through the different examples of the concept. In other words, the C.C.P. is a "mirror" of concept formation with age.

The concept analysis of student responses, using critical and non-critical attributes and Van Hiele theory, helps in the understanding of factors which effect student concept formation. The influence of definition on identification of concept examples also receives a new dimension in the light of such an analysis.

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ANALYSIS OF CHILDREN'S DISCUSSIONS OF GEOMETRICAL PROBLEMS WITH THE FRAME-MODEL

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In a classroom experiment primary school children (aged 9 - 10) were asked to find - with the help of some materials - all the possible cube nets. Our primary aim was not to discover whether these pupils were able to solve this problem or not (some of them were); the experiment was part of a project concerned with the description of pupils' mathematical behaviour in terms of models of Cognitive Science. In earlier experiments we had come to the conclusion that the "hypothetical mechanisms" presented by Davis and McKnight (1979), especially the frame-model (Davis 1980), seem to be useful for the analysis of mathematical thinking processes (cf. Hasemann 1981, 1983). But whereas formerly pupils worked out well known tasks (they just had to apply concepts and procedures which they had learnt earlier), in this new experiment they were confronted with a completely new problem.

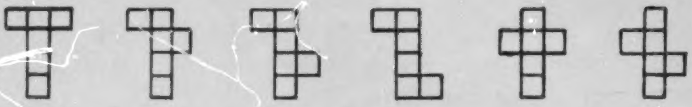
The Experiment

In this experiment we presented cubes and cube nets to primary school children. In a first step the children learnt what "cubes" and "cube nets" are. Then they had to find more and more different cube nets. In a third step - when the children believed that they had already found all of them - they were asked to explain why the cube nets which they had found were "all". The experiment was extended over four lessons; it took place in three primary schools in Osnabrück. The lessons were video-taped (by two cameras).

The third lesson was the most interesting for the experiment: Pairs of pupils were asked first to find "all cube nets with four squares in a row". To help solve this problem they received two types of materials: (i) Six wooden squares by the help of which they could look for more and more figures with six coherent squares; and (ii) cardboard models of cube nets (which they got when they had confirmed themselves that a figure in fact is a cube net). These cardboard models not only served for the comparison of new nets with those which had already been found, but also marked the number of cube nets which a pair of pupils had found.

In each class several pairs of pupils were supervised (interviewed) by a teacher and video-taped during their problem-solving process; later on we transcribed the main parts of their discussions, especially their arguments and reasons why their cube nets were "all". (In the fourth lesson the children searched for the remaining nets; for more details of the experiment see: Hasemann 1982.)

In this paper I shall restrict myself to some results taken from the third lesson. In fact, nearly all the pairs of pupils found all six cube nets of this type in a very short time (in about 5 to 10 minutes):



(Fig. 1)

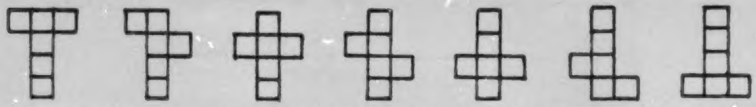
Most of them now argued: "There are no more cube nets because
- for a long period we tried to find more, but we could not; or
- the other pairs found six nets just as we did; or
- the teacher seems to have no more cardboards."

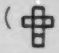
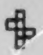
But nearly all the pupils realized the weakness of these arguments. Most of them now changed the question and demonstrated that they could re-construct by use of the wooden squares all the cube nets which they had already found (and which were represented by the cardboards). Some pupils actually believed they had found a final solution to the problem by this procedure. But others tried to show in a systematic way that they had taken into account all the figures which can ever be arranged by six coherent squares (with four squares in a row); i.e. they tried to establish their six cube nets as "all which are possible". For example, let us look at "Marion and Ute":

These two girls had very soon found the six nets, and they worked hard to explain why these nets were "all":
Ute: "It's so hard to explain."

Teacher: "But that's what you should do." (The girls groan.)
Marion: "First, here we have this ladder down..."

Ute picked up this idea immediately, and some time later the girls demonstrated what was meant: By the word "ladder" they meant this order of figures:


(Fig. 2)

The girls did not realize at first that two nets ( and ) - whose cardboard models were on their desk - are missing in this arrangement. But later on they expanded their system in such a manner that the "ladder" was part of it (and, in fact, by the help of their new system one can prove that there are not more than six cube nets!).

Discussion

On the video tapes the pupils' "mathematical behaviour" (Erlwanger 1975) is manifested. But we cannot by description alone explain this behaviour, there remain a lot of questions, for example: Which criteria lead the pupils to believe that they have found "all" cube nets? What are the implications of words like "ladder" for problem-solving processes? Another aspect (which was not mentioned yet): Why are there so many misunderstandings between teachers and pupils even in interview situations? To give some answers to these questions, I shall first consider the frame model:

In 1979 Davis and McKnight presented their model of "Hypothetical Mechanisms in Mathematical Thought", especially - with reference to Minsky (1975) - their model of "frames": "The word frame, as we use it here, means a specific information-representation structure that a person can build up in his or her memory and can subsequently retrieve from memory when it is needed" (Davis 1980, p. 170). But Davis concedes that this description "is hardly a definition in any usual sense". In addition, most of the examples and explications given by Davis, McKnight, or Minsky are just as vague. Hayes (1980, p. 46) even found that "it is not at all clear now what frames are, or were ever intended to be."

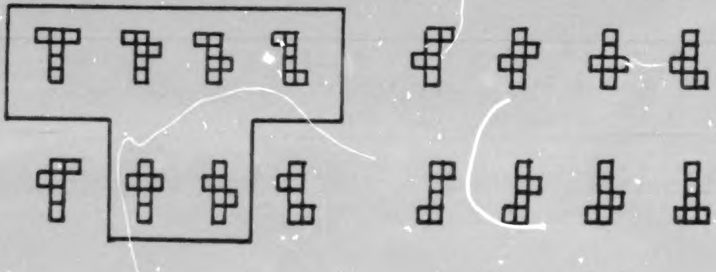
But Hayes gives a new definition. Frames are "representations of concepts": "A frame representing the concept C, with relationships $R_1, \ldots, R_n$, becomes the assertion

$$\forall x\ (C(x) \Leftrightarrow \exists\, y_1, \ldots, y_n \mid R_1(x, y_1) \wedge \ldots \wedge R_n(x, y_n))"$$

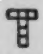
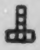
(Hayes 1980, p. 49). Unfortunately, one central intuition behind frames gets lost in this approach, "namely the idea of seeing one thing as though it were another, or of specifying an object by comparison with a known prototype ... This is the basic analogical reasoning ... which Minsky cites as a major influence" (p. 51; by the way: this aspect of "seeing as" seems to have been the first meaning of frames when this notion was introduced in Gestalt Psychology, cf. e.g. Koffka 1935, p. 185: "Orientation, as a factor which determines the shape of our figure, is, then, not an absolute matter, but relative with regard to the frame").

In this paper, I shall use the frame notion as follows: Each person forms his or her own concepts; in the case of mathematical concepts these individual concepts (or "concept images", cf. Vinner and Hershkowitz 1983) might be quite different from the concept definition given for example in the textbook. These individual concepts are represented in frames, i.e. knowing his frames, we are able to speak about a pupil's concepts. Like mathematical concepts, a pupil's individual concepts may be described by a collection of relations or properties; to be precise: The notation of a pupil's frame is an assumption about his individual concept. As the pupil's mathematical behaviour comes from the application of this individual concept, the assumption of specific frames can be tested by comparing his actual mathematical behaviour with the hypothetical behaviour which would result from his hypothized frame.

As an example of a frame let us look at the teachers' frames for solving the problem: "Find all cube nets with four squares in a row": We know that cube nets consist of six squares, whereby there must be one square on each side of the row (it's evident from trying that one cannot form a cube anywhere else). Given these facts, there remain precisely 16 figures which must be tested to see whether they are cube nets or not:

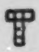
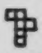
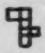
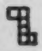


(Fig. 3)

In fact, all these figures are cube nets, but just six are different (those framed, see above Fig. 1). I.e., the teachers' frames might be described by a combinatorial arrangement of figures (which may also be regarded as a sequence of actions varying the figures from  to  ; and of course there are other ways to represent such a system so that it is clear that all possible figures have been taken into account).

From the girls' expressions we learnt that their frames were quite different: Corresponding to the teachers' frames, there is the basic idea of ordering all figures of six coherent squares systematically. But in their frame the figures are ordered in a way which looks like a ladder of descending steps (see Fig. 2).

The word "ladder" and the meaning of this word in every day life seem to have had a strong influence on the construction of this frame in the girls' minds: 1. They saw that there is a relation between their cube nets (represented by the cardboards); 2. they named this relationship, using the word "ladder"; and 3. they generated by use of the ladder-descending-idea all possible figures with six squares. For a moment they were sure they had solved the problem by this procedure, only when the teacher pointed out that two cube nets were missing did they realize that their system was incomplete.

The word "ladder" was used in different ways by the girls and by the teacher, resp.: The teacher denoted by this word a sequence of four figures (, , , ) which belongs to his frame (whereas Ute understood Marion's idea immediately). From the transcript it becomes clear that the girls on the one hand and the teacher on the other were misunderstanding each other for a long period. But they did not realize this because they translated the other's expressions *according to their own frame* (in real classroom situations one can observe misunderstandings like this very often).

Summary

It seems possible to explain parts of pupils' mathematical behaviour with the help of the frame model in a very plausible way. If we are more sucessful in the analysis of pupils' frames, it should also be possible to re-construct their mathematical thought in problem-solving or concept formation processes; i.e. more knowledge about pupils' frames in different fields of mathematics might be a step towards more insight into the structures of pupils' mathematical thought.

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SPATIAL VISUALIZATION
SEX AND GRADE LEVEL DIFFERENCES IN GRADES FIVE THROUGH EIGHT

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Spatial visualization is considered to be one of the two major factors of spatial ability. Spatial visualization is an aptitude that deals with the mental manipulation of rigid figures. Investigating spatial visualization is important because of its relationship to most technical-scientific occupations (Harris, 1981) and especially to study of mathematics, science, art and engineering (Bishop, 1978; Harris, 1981). The spatial visualization aptitude emerges as a component of mathematics ability in most factor analytic studies (Schonberger, 1976), and it shows about the same magnitude of correlation with mathematics achievement as the verbal component does (Fennema and Sherman, 1977). An additional indication that this aptitude is an important consideration is the finding of sex-related differences in spatial visualization. Fennema (1975) indicates that even though the existence of many sex-related differences is currently being challenged, the evidence is still persuasive that in the American culture, male superiority in tasks that require spatial visualization is evident beginning during adolescence. Surprisingly, studies of training programs to increase spatial visualization are very few. Of the existing studies, most have been done with adults and the results are inconclusive. No large-scale study to implement a unit of instruction in spatial visualization for a wide range of grade levels was found. This provided motivation for the study of spatial visualization in grades five through eight reported below (Ben-Haim, 1982).

PURPOSE

The study had two related purposes. The first was to determine existing differences in spatial visualization abilities and in attitudes toward mathematics of fifth through eighth grade students by sex and grade level prior to an instructio al intervention. Research on cognitive variables shows close relationship of the cognitive and affective components, thus investigating performance and attitudes simultaneously seems appropriate. The second purpose was to analyze the effects of instruction in activities involving spatial visualization tasks on the skills and attitudes toward mathematics of a sample of sixth, seventh, and eighth grade students by sex and grade level.

METHODOLOGY

The study was conducted during winter 1982. Data was collected from January 20, to April 20 and included general information on the students and pre-post and retention test scores on spatial visualization tests and attitude scales. There were 1327 fifth through eighth grade students from three sites, urban, suburban, and rural in and around Lansing, Michigan, who participated in the assessment of differences prior to the instruction. Of these, 430 sixth, seventh, and eighth graders (only from site 3 - the rural) participated in the evaluation of the effects of the instruction; of the 430 students, 238 took part in the evaluation of the persistence of the effects (four weeks after the end of the instruction).

The instruments used included a semantic differential attitudes scale (Shumway et al, 1981) and a spatial visualization performance test designed by the Middle Grades Mathematics Project (MGMP), Department of Mathematics, Michigan State University*. The spatial visualization test comprises ten item types relating to different aspects of spatial visualization ability. The Cronbach α reliability coefficients of the test for the various groups of students ranged from .72 to .86. With a sample of 73 students the test-retest reliability coefficient was .787.

The spatial visualization instructional material included ten sequenced activities which required two to three weeks of instructional time. The activities involve representing three-dimensional objects (buildings made from cubes) in two-dimensional drawings and vice versa, constructing three-dimensional objects from their two-dimensional representation. Two different representation schemes are used for the two dimensional drawings. First an "architectural" scheme involving three flat views of the building-base, front view and right view. After students are comfortable with this scheme, they are introduced to isometric dot paper and hence to a representation consisting of a drawing of what one sees looking at a building from a corner. Similarities, differences, strengths and inadequacies of the two schemes are explored.

The math teachers who taught the unit were provided with the instructional material, a teacher guide and a two-hour workshop. The tests and attitude scales were administered during the regular school day by the classroom teachers. The statistical analyses included multivariate and univariate analysis of variance and repeated measures.

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* The MGMP is a curriculum development project funded by NSF-DISE (National Science Foundation Development in Science Education) to develop units of high quality mathematics instruction for grades five through eight. The staff of the project: Glenda Lappen, Director; William M. Fitzgerald; Elizabeth Phillips; Mary Jean Winter; Pat Yarbrough; and David Ben-Haim.

RESULTS AND CONCLUSIONS

Table 1 provides means of the Spatial Visualization Test (SVT, range 0-32) and the Mathematics Attitude Scale (MAT, range 1-5) pretest scores for the entire sample by site by grade by sex. Table 2 includes the P values resulted from the multivariate and univariate analysis of variance for each site. Three planned comparisons were used for the grade main effects at site 3 -- G_1=(5,6) vs (7,8), G_2=5 vs. 6, and G_3=7 vs. 8. No significant interaction was found at sites 1 and 2; at site 3 the only significant interaction was for G_3 by Sex. In addition to the above analyses, a comparison of the data across sites was investigated to determine differences in spatial visualization by site and by sex for students in in grade six.

TABLE 1

Pretest Means of SVT and MAT Scores by Site, by Grade, and by Sex

				BOYS			GIRLS		
		SVT	MAT		SVT	MAT		SVT	MAT
	N	M	M	N	M	M	N	M	M
Site 1 (urban)									
Grade 5	104	7.34	3.552	55	7.25	3.450	49	7.43	3.667
Grade 6	115	8.85	3.099	58	9.05	2.950	57	8.65	3.249
Total	219	8.13	3.314	113	8.18	3.193	106	8.08	3.442
Site 2 (suburban)									
Grade 6	274	12.15	3.200	150	12.91	3.259	124	11.23	3.343
Grade 7	153	12.90	3.027	71	14.07	3.039	82	11.88	3.017
Total	427	12.42	3.297	221	13.28	3.188	206	11.49	3.213
Site 3 (rural)									
Grade 5	94	8.81	3.685	48	9.58	3.788	46	8.00	3.577
Grade 6	208	10.17	3.044	104	11.31	3.064	104	9.03	3.024
Grade 7	170	11.17	2.858	90	11.76	2.696	80	10.50	3.061
Grade 8	209	12.97	2.802	96	15.27	2.836	113	11.02	2.774
Total	681	11.10	3.076	338	12.33	3.007	343	9.89	3.025

It was concluded from the analysis of the data gathered prior to the instructiona intervention that there were:

- * Grade level differences in spatial visualization performance (increasing with age) and in attitudes toward mathematics (decreasing with age).
- * Sex differences in spatial visualization performance (favoring boys), but no sex differences in attitudes toward mathematics.
- * Site differences in spatial visualization performance; as the socio-economic status rose, the performance increased.

TABLE 2

Multivariate and Univariate Analysis of Variance for each Site - P Values for Grade and Sex Differences

	Source of variation	Multivariate P<	Univariate SVT P<	 MAT P<
Site 1				
	Grade	.0003	.0225	.0007
	Sex	.1393	.8447	.0483
Site 2				
	Grade	.0020	.2050	.0029
	Sex	.0026	.0013	.5924
Site 3				
	G_1=(5,6)Vs.(7,8)	.0001	.0001	.0001
	G_2= 5 vs. 6	.0001	.0235	.0001
	G_3= 7 vs. 8	.0012	.0006	.4879
	Sex	.0001	.0001	.6897

Table 3 provides means of the pre-and post-test of SVT and MAT scores for the instruction subsample from site 3 by grade and by sex. Table 4 includes the univariate P values resulted from the multivariate and univariate analysis of repeated measures. Two planned comparisons were used for the grade main effects-(7,8) vs. 6 and 7 vs. 8. Table 5 provides means of the post-and retention-test of SVT scores for the retention subsample by grade and by sex.

TABLE 3

Pre-and Post-test Means of SVT and MAT Scores for the Instruction Subsample by Grade and by Sex

		SVT		MAT	
		Pretest	Posttest	Pretest	Posttest
	N	M	M	M	M
Grade 6	108	11.04	16.96	3.094	3.153
Boys	54	12.48	18.83	3.156	3.265
Girls	54	9.58	15.09	3.032	3.040
Grade 7	142	11.22	20.02	2.918	2.947
Boys	74	11.74	21.00	2.761	2.815
Girls	68	10.64	18.96	3.089	3.091
Grade 8	180	13.23	20.56	2.779	2.737
Boys	79	15.98	23.06	2.816	2.753
Girls	101	12.11	18.60	2.742	2.725
Total	430	12.01	19.48	2.904	2.911
Boys	207	13.48	21.12	2.889	2.909
Girls	223	10.65	17.86	2.918	2.913

TABLE 4

Analysis of Repeated Measures - P Values

	Grade by Sex by Time	Sex by Time	Grade by Time (7,8)vs.6	Grade by Time 7 vs.8	Time effect
	P<	P<	P<	P<	P<
SVT	.5388	.2927	.0002	.0057	.0001
MAT	.5508	.7415	.3160	.3150	.8195

TABLE 5

Post- and Retention-test Means of SVT Scores for the Retention Subsample By grade and by sex

	N	Posttest M	Retention M
Grade 6	52	17.11	18.65
Boys	26	18.85	19.50
Girls	26	15.70	17.81
Grade 7	79	20.06	20.71
Boys	41	20.93	21.46
Girls	38	19.13	19.89
Grade 8	107	21.23	22.36
Boys	45	24.40	25.02
Girls	62	18.94	20.42
Total	238	19.95	21.00
Boys	112	21.77	22.44
Girls	126	18.33	19.72

It was concluded from the analysis of the pre-post data that after the instructio

* Sixth, seventh, and eighth grade boys and girls performed significantly higher on the spatial visualization test; however, no change in attitudes toward mathematics occured.
* Boys and girls gained similarly from the instruction, in spite of initial sex differences.
* Seventh grade students, regardless of sex, gained more from the instruction than sixth and eighth graders.

In addition, the retention of the effects of the instruction persisted; after a four-week period, boys and girls performed higher on the spatial visualization test than on the posttest.

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SPATIAL REASONING: STAGES OF DEVELOPMENT

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Having the ability to visualize in three-dimensional space is a desirable asset for studying mathematics, but we have only the vaguest notions and understandings of how this ability develops. Indeed, even though researchers feel that they know what spatial ability is, and even though spatial ability has been the subject of literally hundreds of research studies

1. no factor analytic characterization of spatial abilities has yet been found (Guay, McDaniel and Angelo, 1978);
2. we have no idea why boys, as a whole, perform better on spatial reasoning tasks than girls (Fennema and Sherman, 1977);
3. we do not know why races, as a whole, perform differently from one another (Mitchelmore, 1980); and
4. we can not even answer with certainty whether spatial ability is inborn or learned (Vladimirskii, 1971).

Although several excellent reviews of the literature have recently been given by McGee (1979), Bishop (1980), Eliot and Hauptman (1981) and Clements (1983), the bottom line remains: we really do not know much about spatial reasoning ability, but we consider it a desirable trait to possess.

Being educators, we take as an underlying hypothesis that spatial reasoning, loosely defined as the ability to formulate and manipulate mental images (Hebb, 1972), can be fostered in a serious way. We can, through teaching, nurture in students the ability to visualize objects and, more specifically, to visualize mentally three-dimensional objects which are presented to them by means of two-dimensional graphical representations.

When presented with the drawing in Figure 1, for example, a person can either see a plane triangle with three lines emanating from its vertices meeting in point P, or he can immediately see a three-dimensional pyramid of which P is the vertex. The context in which the figure appears should dictate to the person, which interpretation to use, but both should be absolutely natural to him. The mental flexibility implied by the choice between the two interpretations is a central aspect of spatial ability. We believe this ability is important to possess, not only for its own right, but because the type of mental processes involved in effecting such visualizations are necessary for and can transfer to other areas of mathematics and science. To mention only a few examples: Linear algebra, real functions of two real variables, propagation of plane or spherical waves, atomic or molecular structure, and so on. Because of this central role which spatial visualization plays in our culture, a program has been designed which is expected to eventually lead to materials for schools.

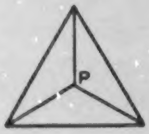


Figure 1

OVERVIEW

The program is planned in several distinct phases. Phase I is concerned with ascertaining the status of children's spatial ability and determining how this ability develops over time. Of interest here is a careful assessment of the type of skills the children possess. Two types of tasks have been chosen as representative for a large class of spatial abilities: Counting analyses of spatial structures given by plane representations and tasks requiring mental folding or unfolding of surfaces. Children of various ages and academic levels are being tested on these tasks by questionnaires and in structured interviews. The aims of the testing are twofold:

1. to get bench-mark data on these question types for a reasonably large section of the student population and
2. to identify cognitive difficulties students have with such questions.

A first impression of the development of spatial reasoning vis-a-vis maturation can be gained on the basis of the data obtained from the questionnaire. But in order to provide a more solid foundation for examining stages of development of spatial reasoning, the same children will be tested each year over the next several years. It will be determined what progress children make in spatial reasoning without specific intervention, since they are not taught spatial topics explicitly.

On the basis of the results from phase I, an instructional sequence will be developed in phase II of the program. The sequence will be composed of several small units. Each unit will consist of an instructional part in which models will be used and a structured series of exercises. One of the goals, for example, will be to enable the students to see and draw intersections of bodies such as the orthogonal intersection of two right circular cylinders of equal radius.

As an earlier study showed (Dreyfus and Eisenberg, 1983), some mathematics teachers are not sufficiently proficient in spatial reasoning to teach it in a serious way. The teachers who participated in that study were junior high school mathematics teachers, but there are indications that their ability on these tasks is representative of the ability of mathematics teachers in general. It is therefore planned to involve a sizable number of teachers in the development phase for two purposes: Build up the teacher's background in the area and get feedback based on their experience with the materials in development.

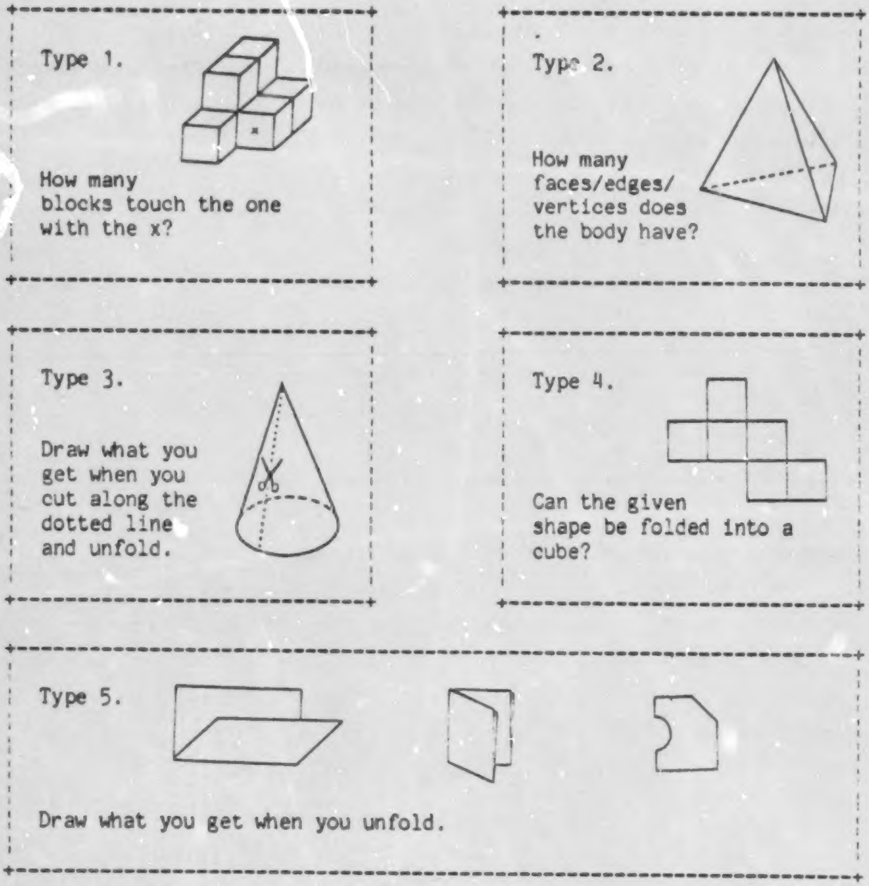
Phase II will be linked to Phase I insofar as the units will be carefully adapted to the stages of development identified in Phase I and will systematically take into account the cognitive difficulties which appear to be common to a large section of the population. Phase III will be concerned with the implementation and the evaluation of the resulting materials.

THE LONGITUDINAL ASSESSMENT

The first stage of the longitudinal study referred to above as Phase I will be described here in detail.

Questionnaire

Two different versions of a questionnaire were written. Each version contained 17 exercises, each question belonging to one of the five question types illustrated in Figure 2. More specifically there were 3 questions of type 1, 3 questions of type 2, 4 questions of type 3, 4 questions of type 4 and 3 questions of type 5.



Type 1.

How many blocks touch the one with the x?

Type 2.

How many faces/edges/vertices does the body have?

Type 3.

Draw what you get when you cut along the dotted line and unfold.

Type 4.

Can the given shape be folded into a cube?

Type 5.

Draw what you get when you unfold.

Fig. 2: Typical Problems

Questions of type 1 and 2 belong to the category of counting analyses in three-dimensional bodies given by two-dimensional drawings and questions of types 3, 4 and 5 belong to the category of mental folding or unfolding of two-dimensional surfaces in three-dimensional space. These question types were chosen because it was felt that they represented two visualization skills representative for a very large class of such skills. Sociological and personal data as well as questions on the students' self-perceived abilities in school mathematics and in spatial ability completed the questionnaire.

Population

The questionnaires were distributed to 244 individuals in two geographically different areas in Israel. They were administered in group and in individual settings. The distribution of the population by grade and sex is shown in table 1.

TABLE 1

Population by Grade and Sex

Grade / Sex	4	6	9	10	Univ.	Total
Female	17	33	24	12	8	94
Male	16	32	8	34	60	150
Total	33	65	32	46	68	244

The University students in the study were all taking an intermediate level course in beginning calculus. Each had passed a nationwide University matriculation exam or its equivalent. For each group the questionnaires required 30 - 40 minutes to complete.

The findings and interpretations of the data collected will be presented at the conference. Particular attention will be paid to pinpointing areas in the data which suggest specific pedagogic intervention.

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CAN WE TEACH HEURISTIC STRATEGIES ?

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A lot of research work in the field of problem solving has been done during the last years. The explored domain turned out to be the more complex, the more intensively it has been invaded.

It is extremely dubious whether general heuristic strategies can be trained. We know that every sufficiently specified strategy, useful in typical tasks, does not lead to success in other situations, which are specific, too. What seems to be possible, and what is necessary at any rate, is training specific strategies, i.e. strategies to solve problems belonging to a relatively narrow topic field and fitting to certain task types, however avoiding restriction to mere routine tasks. After having made experiences with rather narrow strategies, pupils may combine them to more extended strategies, and to generate superordinated strategies; that means those selecting strategies of lower order and estimating them at their efficiency in more complex problem situations.

Parts of an instructional sequence shall be sketched, in which training of content-specific and problem type-specific strategies was attempted, and a report shall be given about difficulties and success in applying them, and about formation of higher strategies. Since the beginning of the term 1981/82 I have been giving mathematics lessons in two grade 7, now grade 8 classes of a junior high school (a german Gymnasium). By several reasons only a small amount of lesson time could be used for the subject in question. So, it was indispensable to concentrate upon topics easily to be incorporated into the curriculum, and admitting an expansion towards the intended aims. Under these conditions geometry seems to be the most adequate subject area in junior high school curriculum to realise the mentioned purpose. The competing area, namely arithmetic and algebra, mediates mainly techniques, and the few problems in which these are applied are too inhomogenious as to train content-specific strategies. In geometry we can distinguish between several types of problems or tasks, above all computation problems, construction problems, and demonstration problems (cf. Holland, 1982, pp. 190-253). Roughly spoken computation problems have the most unique solving techniques, and we will find a nearly unlimited number of exercises of the same type. The latter pertains to construction problems, too, whereas solution techniques vary in grade of difficulty. Demonstration problems allow a tolerably unique solution technique only by a certain kind of selection and grouping of the theorems to be dealt with. Besides that, proving and making demonstrations need non-routine elements to a higher extent than the other mentioned task types, at least with respect to junior high school. Therefore, demonstration problems particularly suggest the inquiry of formation and stabilisation of problem solving strategies. This process must be based upon the acquisition of a "schema" by the pupils, which can be developed further, expanded, and refined (cf. Skemp, 1979).

Such a "schema" may consist in the awareness that a new linking idea has to be added to the premise and the conclusion of a theorem, connecting them logically, and, at the same time, to get at least a rough idea of how that can be done, and whether an invented proof or a given one is sufficient. Such a general and basic pattern might be formed by analysing given demonstrations or those worked out together with classroom fellows, reflecting upon its parts, and upon what it fulfils.

In order to understand demonstrations, and even more, to learn to design demonstrations on their own, pupils need

1. a stock of elements or "bricks", that means parts of formerly studied demonstrations, stored in memory and available to be combined to new demonstrations,

2. an estimation of whether a given or a found demonstration is correct, or whether it contains gaps.

The strategies to be learnt consist of the acquaintance with an instrument and at the same time of a method for its use, f.i.

- the knowledge that a figure can be decomposed by adding one or several auxiliary lines, and together with this the attainments of the properties of special auxiliary lines, such as angle bisector or perpendicular bisector of a line segment,
- the knowledge that equal measure of two line segments or two angles can be demonstrated by using a pair of congruent triangles, and together with this the attainments of the congruence theorems.

Communication demands a formulation of the strategies in language medium. A check-list consists of the following questions:

Which auxiliary lines are under consideration ?
Which partial figures are generated by drawing these line(s) ?
Can I find a pair of congruent triangles ?
How can I point out the congruence of the triangles in question ?

Geometric proofs being based upon congruence transformations might be regarded as an alternative to the method using congruent triangles. There is no doubt that in many a case transformations make visible a deeper ground for a geometric statement. But with respect to uniformity of proof techniques the method of congruent triangles seems to be superior to the transformation method. A stock of stipulated skills must be the basis upon which higher strategies of a hierarchically structured system of heuristic instruments might arise.

The congruence theorems themselves can be interpreted as summarizing conditions for the solvability of construction tasks. They form, together with some properties of the rectangle, the deductive base for the further theorems. It should be mentioned that, from an axiomatic point of view, this base is quite incomplete; in the above mentioned sequence no existence theorems (f.i. for the midpoint of a line segment), no theorems about measurement (f.i. in the case of decomposing an angle or a line segment), no theorems about betweenness or linear order upon a straight line, no theorems about uniqueness have been formulated.

Angle sum theorems and theorems about special types of triangles were taken as an opportunity to speak about proving and about the form of deduction. Above all, properties of different types of quadrilaterals were used to apply and to train the learned strategies.

The report about the outcomes shall be arranged in two parts. At first impediments and difficulties which occured shall be shown by examples. They make obvious lack in mastery of the strategies to be attained. As the main reasons for not finding the solution of a demonstration problem or for giving a wrong solution there can be identified:

1. insufficient treatment of the topics necessary for understanding or solving the problem, or lack in exercise, such as

- using unsuitable auxiliary lines

- attributing too many properties to an auxiliary line, f.i. regarding the bisector of an angle also as the bisector of another one (without proving the latter),
regarding the bisector of an angle also as the perpendicular bisector of a line segment

- applying an auxiliary theorem without checking whether it can be applied in a special situation

- not proving a required property of a given figure generally, but only for a special case

- measuring instead of proving, or confusing parts consisting in measuring with parts consisting of demonstrations

- circular procedure
certain properties of a geometric figure being mentioned or formulated twice within a proof, i.e. enumerated without any proof and, shortly after, deduced

- non-separating a theorem from its converse the pupils denying the necessity to prove the converse of an established statement at all, or not recognizing that in a certain context the converse of a theorem has to be applied, and not the theorem itself

- misunderstanding the meaning of a variable
f.i. confounding the use of a symbol such as "α" to denote angles of equal measure with the use of the same symbol to denote the angle at the bottom on the left;

2. lack of transfer or generalisation, interference by competing principles or strategies,
such as

- decomposing a quadrilateral into two partial triangles instead of four, the decomposition into two triangles having been repeatedly a suitable strategy in the past

- using permanently symmetric parts of a figure even in cases when only one part is sufficient or the two symmetric parts are not congruent to one another;

3. inadequate structuring of the field of perception, or fixation upon certain elements of the problem situation,

such as
- hesitating to formulate the congruence of two triangles, if these overlap, or if they have a common side, so that this side should be used twice

- favouring or even being fixated upon horizontal and vertical auxiliary lines.

Secondly some spontaneous attempts - i.e. not explicitly required attempts - towards the formulation of "higher" strategies are reported. One field of exercise which might challenge the rise f.i. of backward strategies is the treatment of properties of different quadrilateral types. Here, trying to prove not directly the defining property of a figure is near at hand, and to aim at proving such a property, from which the defining property follows. It was not intended to deal explicitly with higher strategies before the pupils would have ample experiences with the mentioned basic strategies. Yet, in some pupils' remarks explaining their own procedure backward strategies are recognizable very distinctly.

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EUCLIDEAN GEOMETRY FOR AVERAGE ABILITY CHILDREN

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The controversy about geometry in (junior) high school has been going on for tens of years (Fletcher, 1971). Indeed, if students don't even grasp the nature of proof (Fischbein and Kedem, 1982), why teach them Euclidean geometry? As a result of the 'Euclid must go' syndrome a wealth of ideas for alternative geometry courses has been proposed (see, for instance, Freudenthal, 1973, chapter 16). Another opinion holds that, because of its intrinsic value, Euclidean geometry should continue to be taught, and it apparently does continue to be taught in a majority of geometry courses (Gattegno, 1980). It is therefore imperative to analyze the cognitive difficulties of children learning Euclidean geometry and to develop teaching strategies designed specifically to deal with these difficulties. One attempt to do this is described in the present paper.

COGNITIVE DIFFICULTIES

Texts for Israeli junior high school mathematics courses are prepared in three parallel series, corresponding to three ability levels: high ability (A), medium ability (B) and low ability (C). Junior high school mathematics curricula typically contain approximately 100 lessons of geometry (out of a total of approximately 400 mathematics lessons). Usually geometry is taught in grade 9. The aim of the geometry curriculum is to acquaint the students with a deductive mathematical structure, a secondary goal being the knowledge of the properties of plane geometrical shapes, which the children know from elementary school. This geometry course builds a deductive chain of axioms and theorems. The accompanying problems deal with building proofs, computations in geometrical figures and constructions with compass and straight edge. This course will be called the 'Standard Course' in the sequel.

Classroom observations during the standard course revealed that its approach does not correspond to the capabilities of average ability students. By this we mean B-level students generally as well as A-level students in schools with a high percentage of socially disadvantaged children. Each student in Israel is labelled by the Ministry of Education as to whether he is socially disadvantaged or not. This labelling is based on a socioeconomic formula that takes into account a variety of factors in the child's environment. Characterisic difficulties of these children in the cognitive and affective domains have been described by Hershkowitz (1980). Some difficulties relevant to the learning of geometry are the language problem, the need for constant success and the inability to organize a complex task. In fact, in order to prove a statement given in a verbal fomulation, the student has to identify assumptions and conclusion, translate from verbal into mathematical language, and find a chain of logical steps leading from the assumptions to the conclusion, whereby the logical order may be different from the order in which he found the steps. The need for translation and the complexity of the task will usually prevent immediate success, and students may therefore fail at such problems or not attempt their solution at all.

At the end of the standard course most B-level students and socially disadvantaged A-level students were unable to independently carry out even simple proofs and largely failed to understand the basic logical principles underlying the deductive structure they were taught. Since it appears that teaching an explicit unit on logic is not an effective remedy (Deer, 1969), it was decided to carry out a detailed analysis of student's difficulties in organizing their thoughts and in building the logical arguments. This analysis revealed that the difficulties were due to a lack of understanding of the following fundamental features:

1. Even 'obvious' statements have to be proved.
2. A statement is correct only if it is correct in every case.
3. Concluding results from given data.
4. Distinguishing the assumptions from the conclusion.
5. Difference between a theorem and its converse.
6. Identification of basic shapes in complex figures.

STRATEGIES

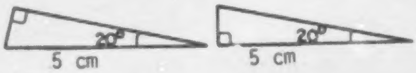
On the basis of this analysis, a revised version of the geometry course was written. The principles guiding the writing were that no essential change in content should be made, but that the approach would be completely changed so as to acquaint the students with the structure of proofs (without demanding them to carry out proofs independently) and, at the same time, to stress the six fundamental features listed above. Methods were developed to deal with each one of them and were carefully and systematically implemented in the text. Two of these methods are described here in detail.

Even 'obvious' statements have to be proved

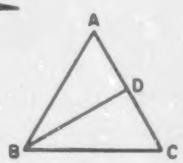
The principle to be discussed is that it is not allowed, in a proof, to rely on figures but that each statement has to be based on statements proved earlier. This principle is implemented in two ways:

a) A statement is presented to the student together with a figure which either misleads or does not allow any conclusion with respect to the statement.

Example: Determine whether the two triangles are congruent:



Example: Given $AB = AC$
and $BD \perp AC$
can you conclude that $AD=DC$?



Remark: Many mistakes occur because the students draw a more regular figure than the given one and draw conclusions from it.

b) Statements which are neither obvious nor easily accessible to intuitive reasoning are emphasized. In such statements the proof is the only basis for believing that the statement is true.

Example: The sum of the external angles of a polygon equals 360 .

A theorem has no exceptions

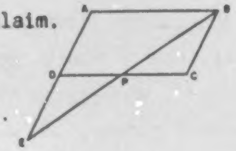
The principle to be discussed is that a mathematical statement is said to be correct if and only if it is correct in each and every conceivable case in which the assumptions are satisfied. To drive the point home students have to correctly complete statements such as

Example: A quadrilateral with four right angles is a _______.

Remark: Many students - and quite a few of their teachers - are hard to convince that 'square' is an incorrect completion.

From the principle it follows that a proof has to be general, whereas for a refutation an example suffices. This point is stressed with exercises such as

Example: Determine whether it is warranted to conclude that △BPC ≅ △EPD and choose an appropriate way to confirm your claim.



Given: ABCD parallelogram

DP = PC

Proof	Reason	Counterexample
___ = ___	___	
___ = ___	___	
___ = ___	___	
△___ ≅ △___	___	

Remarks: - Such an example would typically be accompanied by a similar one, in which the givens don't warrant the conclusion (replace, for instance, 'DP=PC' by 'BP bisects angle ABC').

- The frame provided for filling in the proof helps to prevent frequent mistakes in proving: The illicit use of the conclusion in the argument and the addition of irrelevant information.

EVALUATION

The team writing the new text was interested to receive feedback from the field about the following question: Do average ability students using the new text answer more correctly than those using the standard text to questions pertaining to the deductive structure of geometry? It was hypothesized that such an effect would be observed because of the stress on logical principles in the new text. Additional points of interest were whether differences would be observed on:

1. the facts of geometry (Hypothesis: no difference),
2. general logical reasoning (Hypothesis: no difference),
3. the attitudes to geometry as a school topic (Hypothesis: There will be a difference).

It was decided to carry out both, a formative and, independantly, a summative evaluation of the text.

A draft version was taught, in a first stage, in 6 classes with select teachers, all of them collaborating to some extent with the writing team. A regular feedback mechanism was set up and provided one part of the formative evaluation of the text, the other part being provided by a set of common geometry questionnaires to all classes. As a result of this formative evaluation a series of improvements were introduced in the text yielding the experimental version. In a second stage, an independent evaluator was added to the team and, in a controlled experiment, the experimental version was taught to 23 grade 9 classes, distributed in 15 different schools. Most of these classes (15) were B-level classes in schools whose percentage of socially disadvantaged pupils was medium to low. The remainder were A-level classes in schools with a high percentage of socially disadvantaged pupils. A similar set of 13 classes, being taught with the standard text, served as control group. Except for an initial meeting, in which the principles of the approach and the goals of the evaluation were explained to them, the teachers worked independantly of the developing and evaluating teams.

Six questionnaires were given to the classes in the course of the school year:

a) A geometry pretest checking knowledge on basic notions such as types of angles, triangles and quadrilaterals, as well as simple area calculations.

b) A pretest checking logical abilities on non mathematical topics. A sample question from this test was:

> It is known that 'firehand' is a particular kind of coral.
> 'Firehand' absorbs minerals from the sea.
> Can it be concluded that all corals absorb minerals from the sea?
> Answer 'yes' or 'no' and explain your answer.

c) A midyear geometry questionnaire containing ten problems on the curriculum. The problems were carefully formulated in order not to introduce any bias in favour of either group. The majority of the problems required identifying the correct conclusion (4x), justifying a given conclusion (4x), or doing a geometric computation. Some questions also dealt with negation (4x), counterexamples (2x) and a converse theorem (1x).

d) A semantic differential type attitude questionnaire comparing attitudes to geometry with those to algebra (same teacher) and bible studies.

e) A logic posttest, analoguous to (b).

f) A geometry posttest, analoguous to (c).

RESULTS

The results of the evaluation essentially confirmed the hypotheses stated above. They showed that the experimental group, using the new text improved much faster than the control group. Most of this advantage of the experimental over the control group was stable. The areas in which the experimental group performed better than the control group were the justification of conclusions, the identification of false statements and their refutation by counter examples. No difference was found in the logical abilities of the students outside of geometry. More detailed findings will be presented at the conference.

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WORKING BACKWARDS IN SOLVING GEOMETRIC CALCULATION PROBLEMS

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1.PRELIMINARY REMARKS

In 1980 at the University of Giessen a project was initiated,in cooperation with some nearby Secondary Schools,to investigate the possibilities of problem-solving training using geometrical calculation problems.

The first teaching experiment was conducted in 1980 with 113 grade 8 students of levels A to C.(In German Comprehensive Schools the students are usually classified according to their performances in mathematics in courses of level A,B and C with decreasing standards from A to C.)

Under normal classroom conditions the students worked for two weeks on calculation problems concerned with the measurement of angles. The two examples in Fig.1 are taken from the post test.

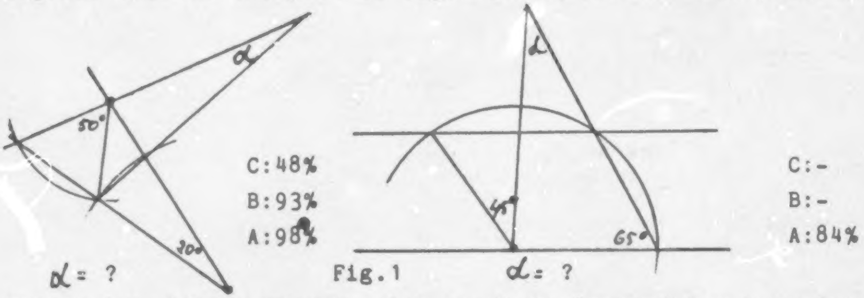


Fig.1

With help of some examples the students learned how to solve problems by working forwards but they did not get any systematic instruction regarding strategy. The findings of the experiment are discussed in full detail in Holland(1980).

But to attack a calculation problem the strategy of working forwards is certainly not always the appropriate method.

We therefore initiated a further classroom experiment to get answers to following questions:

(1)To what degree is the method of working backwards in solving calculation problems teachable to students of grade 7 to 10?

(2) Is a transfer observable from the trained problem class to other classes of geometric calculation problems?

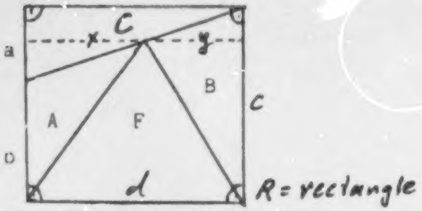
(3) How much do students make use of the method of working backwards if they have learned this method before as well as the method of working forwards?

(4) What special difficulties arise if students try to solve a calculation problem by working backwards? What kind of errors can be observed?

To give the empirical investigations a theoretical background a computer-simulation model has been constructed which is capable of simulating the strategical aspects of the problem-solving process, namely the strategies of working forwards and working backwards. In contrast to J.G.Greeno's sophisticated and comprehensive computer model PERDIX (c.f.Greeno 1978) our model does not simulate pattern recognition and propositional inference. This ommission is justified because our model is - to use Greeno's words -more prescriptive and developmental than descriptive and analytic. The computer,which solves a find-problem by working forwards or working backwards, is a valuable model of those procedures which are necessary to perform the task. It informs the teacher of possible errors and difficulties which are to be expected and supplies him with instructional remedies.
A short outline of the model is given in 4.

2.THE EXPERIMENTAL DESIGN

The experiment was conducted during the first half of 1983 with grade 8 students from two Secondary Schools near Giessen. From the total of 95 students who took part in the experiment 50 were in an A-course and 45 in a B-course. During the first two weeks subject and setting were similar to those in our first experiment. The students solved calculation problems concerning measurement of angles by working forwards. Subsequent to a test and during a further period of two weeks the students were trained in working backwards with calculation problems concerning measurement of polygons. A typical example together with a standard solution is given in Fig.2.



given: $a = 4cm$
$b = 6cm$
$A = 12cm^2$
$B = 15cm^2$
unknown: F

goal	arg
F	A, B, C, R
C	a, d
d	x, y
x	b, A
y	c, B
c	a, b
R	c, d

goal	arg	formula	measure
c	a, b	$c = a+b$	10 cm
x	b, A	$x = 2A : b$	4 cm
y	c, B	$y = 2B : c$	3 cm
d	x, y	$d = x+y$	7 cm
C	a, d	$C = (a \cdot d) : 2$	$14 cm^2$
R	c, d	$R = c \cdot d$	$70 cm^2$
F	A, B, C, R	$F = R-A-B-C$	$29 cm^2$

Fig.2

The students had been instructed to fill in the two tables using a sequence of three procedures:

1. Fill in the table on the left using a sequence of working backward steps.
2. Transfer the table on the left to the first two columns of the table on the right, but the rows now ordered in a working forward manner.
3. For each row of the table on the right note down the formula which serves to calculate the unknown measure from the already known measures.
   Carry out the calculation.

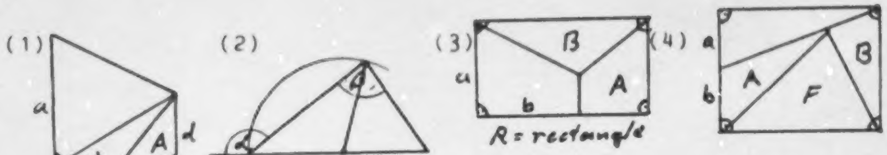
While the two last procedures have a simple algorithmic character, the first procedure is the real core of the problem-solving process. With help of examples the students had been instructed to perform a backward step according to the following procedure:

(1) Select a variable as the new goal and write it down in the first column of the left table.

(2) Look for a set of variables which determinate the goal-variable by applying a known formula to the given geometric figure. If you have found such a set write it down into the second column of the table.

3. POSTTEST AND PRELIMINARY RESULTS

The post test consisted of four problems (Fig,4). Unfortunately to date only the results of 15 students of level A are available.



(1)
a = 12cm
b = 4cm
d = 8cm
A = 12cm
T = ? (trapezium)

(2)
α = 160°
β = ?

(3)
a = 6cm
b = 5cm
A = 15cm
R = 48cm
B = ?

(4)
a = 4cm
b = 6cm
A = 12cm
B = 15cm
F = ?

Fig.3

	1.	2.	3.	4.
solution by working backwards	11 (73)	5 (33)	4 (27)	3 (20)
solution by working forwards	2 (13)	5 (33)	7 (47)	7 (47)
solution by a mixed strategy	1 (7)	2 (13)	-	-
false or not tried	1 (7)	3 (20)	4 (27)	5 (33)

The students had been asked to find solutions for the first two problems by working backwards. For the remaining two problems they could choose their favorite method. The first problem of medium difficulty was selected to assess the ability of the students to apply the method of working backwards to new problems of the trained problem class. The second problem was selected to assess the transfer of the method to another problem class. The negative result is problably due to the fact that the problem belongs to a problem class to which the method of working forwards had been applied in a preceeding training. The third and fourth problem should give information concerning the degree to which the students prefer the method of working backwards (when they had been trained in this method with problems of the same problem class).

4.A COMPUTER MODEL OF THE PROBLEMSOLVING PROCESS

In our experiment,to obtain the solution of a calculation problem by working backwards the student is expected to fill in the two given tables. As we have seen this task may be accomplished by a sequence of three procedures.Because only the first entails the strategic aspects of the problem-solving process we confine the description of the model in the following to the first procedure. But in filling in the table we allow forward steps as well as backward steps.

We describe the problem-solving process using a state-operator representation. Goalstate is the completed table as a sequence of forward steps or backward steps. The initial state is the empty table. In each step of the problem-solving process a new row is added to the table.Therefore we identify the rows of the table with the operators in our state-operator representation. Since each operator op is an ordered pair of two components we write op=(op.goal;op.arg). The first component op.goal is a single variable whereas the second component is a set of variables. For the human problem solver each operator is itself the result of a process in which an abstract geometric theorem or formula is applied to the given geometric configuration. For example in Fig.2 the three operators (A;b,x),(b;x,A) and (x;b,A) are the result of applying the formula for the area of a triangle to the triangle with length b,altitude x and area A. As Greeno has demonstrated with his computer simulation model PERDIX this process may be considered as a process of visual pattern recognition and propositional inference(Greeno 1978). In our model the operators are immediately derived from what we call "atomic bricks" of the given geometric configuration. Each atomic brick is the concretization of an abstract formula within the geometric configuration . We represent it by an ordered set of variables. For example in Fig.2 the ordered set (T,d,b,c) is an atomic brick which is generated by applying the formula for the area of a trapezium to the given configuration. Because of T=1/2 d(b+c) each of the four quantities T,d,b,c is determinated by the remaining three. Therefore four operators can be derived from the atomic brick (T,d,b,c).
The set of atomic bricks, which represents the given geometric configuration is stored in the computer from the beginning and supplies the operators for the problem-solution. It must be extensive enough to guarantee at least one solution. Also it is evident that the number of possible solutions which can be found grows with the number of available atomic bricks.

In each step of the solution a new operator is added to the current problem state. This may be done as a step of working forwards or as a step of working backwards. Here we can give only a rough descripton of the procedure for working backwards:

(1)Select the new actual goal from the set of possible goals.
(2)Determinate the set of all b-applicable operators which comply with the condition: op.goal=new actual goal;
(3)If the set is not empty then select at random one operator and add it to the problem state, else try a backtrack.

(The concept b-applicable covers several conditions which an operator must meet to be applicable in a backward step.)
The following is a computer solution of the problem in Fig.2.

Atomic bricks:
Additivity of area:
(R,A,B,C,F)
(T,A,B,F)
(R,C,T)
Additivity of length:
(d,x,y)
(c,a,b)
Area of a rectangle:
(R,c,d)
Area of a triangle:
(A,b,x)
(B,c,y)
(C,a,d)

Solution by working backwards

nr	op.goal	op.arg
1	F	A,B,C,R
2	R	C,T
3	T	b,c,d
4	c	a,b
5	d	a,C *
6	d	x,y
7	x	b,A
8	y	c,B
9	C	a,d

In step 5 the star "*" indicates that this row is invalid as a consequence of a backtrack in step 6. Here the actual goal originally was the variable C. But because no operator was available a backtrack is the consequence.

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E. COGNITIVE STUDIES IN ALGEBRA AND RELATED DOMAINS

1. *FUNCTIONS*
2. *STUDENT CONCEPTIONS AND MISCONCEPTIONS*

REPRESENTATION AND UNDERSTANDING: THE NOTION OF FUNCTION AS AN EXAMPLE

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We rarely mention the word "*understanding*" in our work since it encompasses so much that like an "*iceberg*" it conceals more than it shows. In order to relate understanding to representation, we shall give a few features of understanding.

UNDERSTANDING

a) Understanding can be checked by the realization of definite mental acts. It implies a series of complex activities.

b) It presupposes automatic (or automatized) actions monitored by reflexion and planning mental processes. Therefore, understanding cannot be exclusively identified with reflected mental activities on concepts.

c) Understanding is an on-going process. The construction of a ramified system of concepts in the brain is what brings in understanding. Mathematics concepts do not start building up from the moment they are introduced in class by the teacher. This well-known tenet is not easily nor often put into practice in day-to-day teaching.

d) Several researchers attempt to determine stages in understanding. We incline to believe that understanding is a cumulative process mainly based upon the capacity of dealing with an "*ever-enriching*" set of representations. The idea of stages involves a uni-dimensional ordering contrary to observations.

REPRESENTATIONS

We think it is worth making a distinction between representation on the one hand and symbolism or illustration on the other. Let us use DAVIS (1982) definition which exemplifies what we mean.

> A representation may be a combination of something written on paper, something existing in the form of physical objects and a carefully constructed arrangement of idea in one's mind.

A representation can be considered as a combination of three components: 1) symbols (written) 2) real objects and 3) mental images. We believe however that verbal or language features are equally predominant since they are the links in between those elements. We assume also that one can find representati without a real object component. In order to show that such a subtle and intricate distinction brings about some pay-off we shall use it with the concept of function.

WHAT IS A FUNCTION?

Can a single definition encompass the rich meaning of such a notion. In order to simplify the algebraic or formal treatment of this concept in an axiomatic framework, contemporary mathematics has divided definitions of function based on the notion of cartesian product. For example, it can be viewed as a triple of sets (A, B, C) where $C \subset A \times B$ such that if (a_1 , b_1) and (a_1 , b_2) belongs to C than $b_1 = b_2$.

SEMANTIC DOMAIN

The idea of representation helps us in distinguishing several facets of the concept of function. With FREUDENTHAL (1982), we believe that behind the general idea of function lies many basically different objects. Freudenthal uses a term equivalent to phenomenological status while we prefer the expressio semantic domain. In fact, we both claim that even though we can define transformation, variable, sequence, permutation, isomorphism within the framework of function. Those notions remain substantially different in the sense that most forms of reasonning involving each of them are substantially different.

Let us scrutinize two semantic domains. (The size of the paper commands not to go farther). When a function is envisaged as a variable, the role of the domain is often played down if not totally disregarded. The domain is implicitely used, necessarily ordered and very often dense. The nature of the variation is stressed. Mental images related to it and typical verbal descriptions are closely connected to the primitive notion of variation and continuity . In fact, we see a variable changing and our concept of variable is this capacity of the mind to characterize this change. The continuous cartesian graph is then the natural illustration of a variable. As History tells us (see YOUSCHKEVITCH (1979)) , the use of a curve seems to be pre-requisite for the construction of this semantic domain and also that it seems to branch out from a rejection of the proportional change as the unique model of variation.

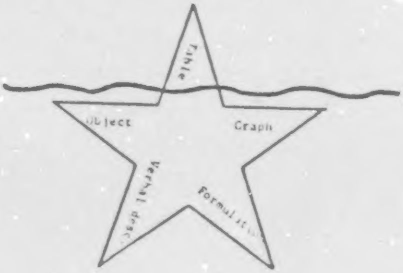
Functions as a transformation require more "intellectual efforts" to deal with because they cannot be conceived without some reference to the domain*. Mental images of the representation of this semantic domain must involve at the same time the domain* and co-domain* (source and image*). Any illustration must suggest the typical essence of a transformation showing initial and resulting objects. The idea of invariant seems to be intrinsically linked with this semantic domain. Geometric transformation other than simple translations or rotations are good examples.

SEMANTIC DOMAIN AND REPRESENTATION

We introduced in JANVIER (1980) the translation skill table which shows translation in between two of the following modes of illustration: graphs, formula, table, verbal description. We insisted on the need for direct translations which are rarely taught in class.

We now believe that this conception must be widened. We think that it is better to use the word "*schematisation*" (which sometimes may be an illustration) and use the word representation with its more general meaning (as in DAVIS (1980)). A translation between schematisation is then performed within a representation.

By analogy, a representation would be a sort of star-like iceberg which would show one point at a time.



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* mathematics meaning

A translation would consist in going from one point to another. This descripti of a representation has the advantage of insisting on the global and "*inseparab* character of a set of schematisations.

At the conference we shall introduce the idea of predominant schematisation and examine how it relates representation and intuition.

A MAJOR DIFFICULTY:
THE CONTAMINATION COMING FROM CLOSE SCHEMATISATIONS

A usual mistake when working within a semantic domain consists in transfering features of one schematisation to another. We shall give three examples relate to the idea of variables.

a) Contamination: verbal ⟶ formula

CLEMENT and KAPUT (1979) tells us about students in difficulty with the following problem: "At a certain university, there are six times as many students as there are professors".

As they write: "25% to 30% of freshman engineering students write $6S = P$. This percentage go to over 50% when a non-trivial ratio is used." In addition to the wrong transfer of the linguistic form "Six times more students" to "6S", it is worth mentioning how "S" becomes student rather than the number of students as "g" is usually used for gram .

b) Contamination: graph ⟶ picture.

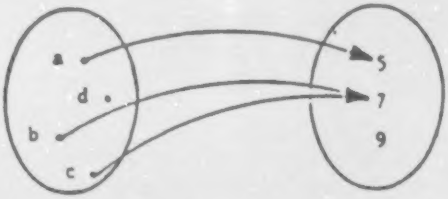
Let us recall briefly the racing-car test item which we introduced in JANVIER (1978). It showed that often pupils tend to see into a graph a total or partial picture of some situation involving the variables with which they deal.

c) Contamination: verbal ⟶ graph
We very often noted that children have difficulty in "*expurgating*" the expressi "*grow fast*" from the idea of being tall it insidiously contains.

CONCLUSION

At the conference, we shall look at other difficulties which can be described within this framework. We shall conclude this paper in suggesting the formidable distance between the richness of the concept of variable and the idea of function as too often presented in textbook.

Here is a function



a) Give its domain, co-domain and range

b) What is the image of b?

c) Draw its cartesian graph

It seems that someone has forgotten something somewhere!

Université du Québec à Montreal
May 1983

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FUNCTIONS - LINEARITY UNCONSTRAINED

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Introduction

The function concept, one of the basic concepts of mathematics, constitutes a major part of the school curriculum in many contemporary programs. "The development of a sound understanding of the function concept provides a solid cornerstone on which to build additional mathematical concepts in later courses" (CUPM 1969 in Buck, 1970).

Undoubtedly, a "sound understanding" of the function concept involves many aspects, from familiarity with the various concepts in its set definition (domain, range and rule of correspondence), to a conceptualization of the function as a relationship between variables. These various aspects and the relationship between them can be found in the historical development of the concept and its definition (Boyer, 1946).

In Israel students first encounter the concept formally in ninth grade mathematics. In the Rehovot program, which is widely used in Israel, they are introduced to a set definition of the function as a many-one correspondence between elements of a domain and a range. The definition and most of the Rehovot program on functions is similar to SMSG and SMP. In parallel, the concept is also used in the study of science, where it is mainly conceptualized as a relationship between variables.

The studies reported here are part of a larger investigation of the understanding of various aspects of the function concept by ninth grade students (Markovits, 1982). In particular this investigation examined student understanding of the set definition in the algebraic and graphical representations, and some aspects of conceptualizing the function as a relationship between variables - the subject of this paper.

Many of the functions that students meet or will meet in their study of science and mathematics, are not defined completely with a domain, range and rule of correspondence, but are given implicitly by constraints the function has to satisfy. These constraints usually do not define a unique function, but an infinite family of functions. For example, in science, the student performs several measurements in the laboratory and attempts to determine the functional relationship between the variables. Even after ignoring measurement errors, it is essential that the student realizes, that in spite of the fact that by some interpolation and extrapolation method, he or she decides upon a certain function, mathematically this is one of many other possible functions. To what extent are students aware of this fact? What kinds of functions will students construct to satisfy given constraints? It is conceivable that the functions they will construct will be strongly influenced by the images they associate with the concept. Thus by asking students to give examples of functions that satisfy certain constraints one can learn both about their understanding of the assumptions behind this important task, and about their image of functions. In particular the prototypical image that the student holds of functions will become apparent (Rosch, 1975).

The first study

The study included about forty items. In each of them, the students were asked to give an example of a function satisfying some given constraints, and to specify how many different such functions exist. The items differ from each other in the following factors, which we thought might influence student responses.

a) Type of representation

In some items the students were asked to give an example in a graphical representation, while in the others (with similar constraints) in algebraic representation.

Examples

1) *In the given coordinate system, draw a graph of a function, which increases throughout the domain.*

2) *Give an example of a function, in algebraic form, which increases throughout the domain.*

b) Type of constraints

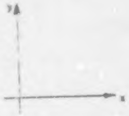
The given constraints were of three kinds:

- discrete constraints - i.e. "points" through which the function has to "pass".

Example 3) *Give an example of a function (in algebraic form) for which $f(3) = 5$ and $f(6) = 8$.*

- properties that the function has to satisfy.

Example 4) *In the given coordinate system, draw a graph of a function, which in part of the domain increases, and is constant in the other part.*



- domain and range of the function.

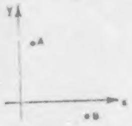
Example 5) *Give an example of a function (in algebraic form) from the real numbers to the positive numbers.*

c) Number of constraints

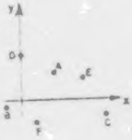
In some items the function was restricted by one constraint only, while in others by a number of constraints.

Examples

6) *In the given coordinate system, draw a graph of a function such that the coordinates of each of the points A, B represent a preimage and the corresponding image of the function.*



7) *In the given coordinate system, draw a graph of a function, such that the coordinates of each of the point A, B, C, D, E, F represent a preimage and the corresponding image of the function.*



All the items had an identical second part:

The number of (different) functions which satisfy the above condition(s) is

- *(1) 0,*
- *(2) 1,*
- *(3) 2,*
- *(4) more than 2 but less than 10,*
- *(5) more than 10 but not infinite,*
- *(6) infinite*

explanation: ______________________

____

Results

Linearity

Most of the responses were restricted to linear functions, and only very few students appreciated the fact that the number of different functions that could satisfy the given constraints is infinite. In the graphical representation almost all functions given were composed of straight line-segments. In the algebraic representation the given rules of correspondence were linear.

The following examples from the responses illustrate the students adherence to the linear function concept. In the third item above, all the students (n = 17) who gave a correct function gave the linear rule $f(x) = x + 2$. Although it may be argued that, in this case, the given constraints "suggest" the linear function, we suspect that the reason is more profound since only about a third of the students said that the number of different functions is infinite, while about half said that there is only one function. The linear image is further emphasized when three constraints were given $f(3) = 4$, $f(6) = 7$ and $f(8) = 13$, where it was impossible to give a linear function. Only one student out of fifty gave a correct function, using the Venn diagram representation, while about half of the students claimed that a function satisfying all three constraints does not exist.

Constraints

In almost all problems, neither the type of representation, nor the kind, nor the number of constraints, had any influence upon the linear feature of student answers, but rather the number of correct answers. In either of the two representations, students did not have many difficulties when the constraints were specified as properties the function had to satisfy. Constraints upon the range and/or domain of the function caused some difficulties, especially in the algebraic representation. However, major difficulties were manifested in both representations, when the constraints were given by "points" through which the function had to "pass", and the number of correct responses decreased as the number of given constraints increased. Although students had difficulties in both representations, the graphical form was relatively easier.

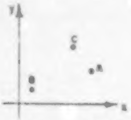
The Second Study

The results of the study regarding students "linear" function images, accord with the results of Karplus (1979). In a more restricted investigation of functionality but with a larger sample (age range 11-18) he also identified the predominance of a "linear" conception of functions. However, the items in his study had a scientific context rather than a pure mathematical one. The question arises how the context of the problems influences student responses in this area. This question was investigated in a second study which examined the effect of context in problems where the constraints were given by "points" on a graph, through which the function had to "pass". This topic was chosen because it was identified as most difficult in the first study. Two questionnaires, which differed in the problem context only (pure-mathematical vs. scientific) were written.

Examples

pure mathematical context

a) *In the given coordinate system, draw a graph of a function such that the coordinates of each of the points A, B, C represant a preimage and the corresponding image of the function.*



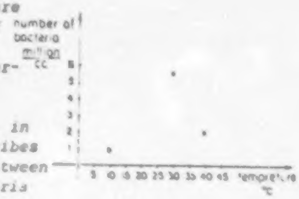
b) *The number of different functions, that can be drawn is:*

(1) *0*
(2) *1*
(3) *2*
(4) *more than 2 but less than 10*
(5) *more than 10 but not infinite*
(6) *infinite.*

scientific context

A Weizmann Institute scientist performed experiments with bacteria culture. In each culture there were certain types of bacteria.
The number of bacteria in a culture depends on temperature and the bacteria type. The scientist performed 4 experiments, each with a different culture composition.

The points in the figure describe the number of bacteria counted in a culture at three temperatures.



a) *Draw a graph which, in your opinion, describes the relationship between the number of bacteria and the temperature.*

b) *The number of different possible graphs that can be drawn is:*

(1) *0*
(2) *1*
(3) *2*
(4) *more than 2 but less than 10*
(5) *more than 10 but not infinite*
(6) *infinite*

The questionnaire included items with 2, 3 and 6 constraints, and a fourth item in which the students had to interpolate and extrapolate. We also wished to investigate the effect of ability on student responses, so the questionnaires were given to high ability students (A stream) and to lower ability students (B stream) in Grade 9.

Results

Independent of the context, the students' concept remained "linear". Most of the functions they drew were composed of straight line-segments, and a few understood that the constraints define an infinite family of functions. The results for the first three items are given in the following two tables.

High ability students

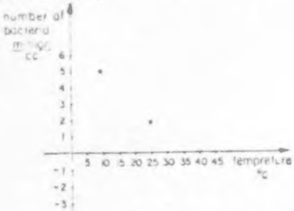
	Percentage of "non linear" examples and of correct examples			Percentage of "infinite" responses		
number of constraints / context	2	3	6	2	3	6
pure mathematical (n = 41)	19.5 (98)	19.5 (88)	19.5 (73)	39	47	41
scientific (n = 42)	14.3 (93)	21.4 (74)	9.5 (57)	43	35	38

Lower ability students

	Percentage of "non linear" examples and of correct examples			Percentage of "infinite" responses		
number of constraints / context	2	3	6	2	3	6
pure mathematical (n = 32)	12.5 (81)	12.5 (50)	6.3 (34)	28	28	28
scientific (n = 30)	6.7 (90)	6.7 (83)	3.3 (75)	30	23	30

In the fourth problem, most students used linear interpolation and extrapolation and overall, in both questionnaires, only 10% of the high ability students and 2% of the lower ability gave a correct answer.

In the "scientific" problem the dominance of the linear image is demonstrated since linear extrapolation leads to an unreasonable (negative) answer.
There were students who resolved the dilemma in some manner (see below), but many students used linear extrapolation blindly.



What can you say about the number of bacteria at 45°C ?

The distribution of student responses for the number of bacteria is given in the following table

	A stream (n = 41)	B stream (n = 30)
One cannot know	12%	8%
Zero	27%	32%
Wrote $-2\ \frac{\text{million}}{\text{cc}}$, but said that the number is zero, because a negative number of bacteria cannot exist	29%	24%
$-2\ \frac{\text{million}}{\text{cc}}$	15%	28%
something else, incorrect	17%	8%

The previous tables indicate that the context had an effect on student responses. Overall, the high-ability students performed better in the pure-mathematical context. A t-test run on a general score which was computed for the three problems yielded a statistically significant difference ($\alpha = 0.03$). The lower ability students performed better in the scientific context, and a t-test run on the same general score, yielded also a statistically significant difference ($\alpha = 0.01$).

The results for the low-ability students are not surprising. They had not encountered problems of this type in mathematics, but had some familiarity with such problems in the scientific context. Low-ability students are known to have difficulty in dealing with completely unfamiliar problems. In fact, those students in this group who received the mathematical problems indicated that they were not clear as to what was required of them.

The high ability students seem to have related the mathematical problems with their mathematical experience and to have used this information effectively. It is possible that such a link was not formed for problems with a scientific context and thus performance was not upgraded by their study of functions in mathematics.

In both ability groups, the context had no effect on the number of students who realized that there is an infinite number of possible functions which satisfy the constraints. In both questionnaires and both ability groups and as in the first study, the more constraints given, the lower the number of correct answers.

As in the first study, this study suggests that students have a mostly linear image of functions, which is not influenced, either by the type of representation, or by the kind or by the number of constraints, and also not by the context.

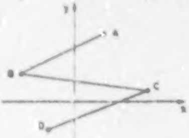
Summary

Two major results emerge from this study. First, many ninth grade students hold a linear prototypical image of functions. Second, many do not appreciate that there may be an infinite number of functions which satisfy a given set of discrete constraints.

The source of the linear prototypical image is quite obvious. This is the simplest function (disregarding the constant function which is not regarded as a function by many students since it does not involve variation). Also, at this stage, the students experience in mathematics and science involves mainly linear functions.

As to the belief that there is only one function which satisfies a given set of discrete constraints, there may be several explanations: (1) Students are used to giving unique answers to problems and have difficulty in accepting lack of closure (Wollman et al, 1979). (2) The linear image that students have, reinforced by their study of geometry (where two points "define" a line and there are many lines through a single point), leads to the unique function satisfying two or more discrete constraints, and an infinite number of possible functions in the case of one "point" through which the function must pass ("there is an infinite number of straight lines through a single point"). The linear-geometric conception influences also the responses to the algebraic items, since there is evidence in student answers that, in many cases, they translate the problems into graphical form.

In addition to the explicit use of geometrical explanations in their answers, there are other facts which support our interpretation. All students who drew curves in the graphical representations, also realized that there are an infinite number of functions and gave a correct explanation. Also, the following type of function (sic), indicating a geometrical approach, was quite frequent.



These results have implications for instruction both in mathematics and in science. Students should be introduced to a larger variety of functions, and the use of the linear function should be de-emphasised. Furthermore, the role of constraints in determining the nature and number of functions should be discussed, both in mathematics and in science, together with an emphasis on interpolation and extrapolation. This is especially important since many teachers are unaware of the predominance of the linear image of functions (Zehavi, 1983).

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INTUITION AND LEARNING OF THE FUNCTION CONCEPT

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THE PROBLEM

A teacher assigns his 10 to 11 year-old 4th-grade pupils the following problem:

> A piece of land is 288 meters long and 72 meters wide. The owner wants to put a fence around it and wants to farm two-thirds of the ground. On each of two sides of the land a gate of 3 meters each is to be installed.

Why did the teacher choose this task?
Are there clear didactical reasons which caused the teacher to opt for this problem?
Which specific solving process - and what learning process - was intended to be initiated in the pupils?
Which is the mathematical structure hidden behind the task which should be discovered intuitively by the pupils?

The didactical analysis shows immediately that there are some complex problem areas:

- the combination of perimeter calculation and area calculation,
- perimeter calculation:
  (1) perimeter (2) considaration of the gates
- area calculation:
  (1) total area (2) one third of this total.

It is obviously expected from the pupils that they are able to solve this complex set of tasks. Two questions, which we will have to deal with, are therefore:

1) How must mathematics courses in the first four years of school practice be structured from a didactical and a

methodological point of view, in order for the children to acquire the capability of doing the necessary analysis of the problem to be solved and arriving at a reasonable structuring of the problem-solving process?

2) Which (cognitive) abilities of the child are challenged when he or she tries to solve such a task and how can these abilities be enhanced and advanced through mathematical instruction in such a way that the children may discover the typical attributes of the function concept?

In answering these questions we will try to analyze the actual, perceivable activity scope of pupils, who have spent a total of about 90 minutes on this task. The actual instruction is recorded on tape and the overall results of the pupils have formed the basis of the following evaluation.

THE PERIMETER CALCULATION AND THE FUNCTION $y \cdot x = 720$

The pupils found 17 different paths of solution for the first aspect of the task - the calculation of the length of the fence. Each individual solution is composed of from 2 up to 7 individual steps. Altogether, 54 arithmetic operations were carried out, i. e. (specifically) 13 additions, 10 subtractions, 22 multiplications and 9 divisions. The pupils suggested - when working without the help of the teacher - three different kinds of solution strategies, which may be characterized as follows:

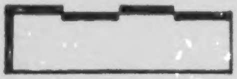
First group
Standard solution by applying the relationsship

perimeter = twice the length + twice the width

Second group
Solution strategies with the help of using half the perimeter

Third group
Solution strategies with the help of using a common divider for length and width

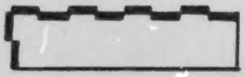
For the solutions of this group a very interesting viewpoint is characteristic. No longer are the figures "perimeter" or "half the perimeter" - figures derived from the factual situation - the guiding principle for the solution. The solution strategy is here largely determined by the connection between length and width of the described rectangle, which is obvious to the pupils.



The length is four times the width. The fence thus consists of 10 such pieces and one piece of these is reduced by the length of the two gates (6 m), since the total reduction for the gates can be effected from only one such part of the pieces. This solution is already considerably far from the "reality", the factual situation of this task, according to which the gates are to be placed at different sides of the piece of land.

A further differentiation within this solution approach comes from the following idea: one could also allocate that part of the perimeter which accounts for the two gates, to the ten individual pieces at an equal share. Then ,- each part is reduced by $\frac{6\text{ m}}{10}$. As a solution, we then arrive at

$$10 \cdot 71{,}4\text{ m} = 714\text{ m}$$



Two other solutions represent a variation of this solution. They use the common divider 36 or 12 of length and width instead of 72.

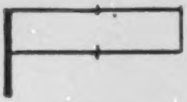
For the common divider 36, there are 8 sections for the long side of the rectangle, two sections for each short side, i. e. 20 sections per 36 meters each in total.

The next solution arrives at 60 sections (6 at each width, 24 at each length side), each measuring 12 meters. In order to account for the gates, the pupil argues:

" 60 times 12 meters, that would be 720 meters; and there I thought: if I take half of a section away from there, and calculate 59,5 times 12 meters, then the gates are subtracted in equal sections (because 12 m times 1/2 are a total of 6 m)".

Thus, the striking aspect of this solution is that the search for number proximity (common dividers or multiples) is not limited to length and width, but is also extended to the length of the gates!

Remarkable variations of this solution approach with number proximities are two solutions, which use as a refence figure twice the width, which, at the same time, represents half of the length. The perimeter consists of five such sections.



If one summarizes the variety of solutions in one table, one can arrive at the following synopsis:

common divider or multiple	number of sections	consideration of gates	comments
72	10	a) -6	corresponds to factual situation (c.)
		b) distribution over 10 sections	does not correspond to factual situation (n.c.)
36	20	-6	(c.)
12	60	a) -6	(c.)
		b) gates are 1/2 section	(n.c.)
144	5	a) -6	(c.)
		b) distribution over 5 sections	(n.c.)

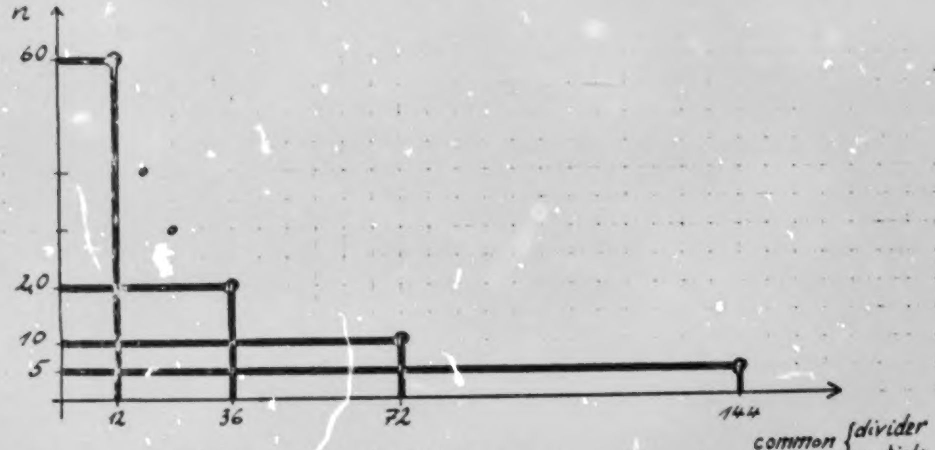
Further solutions with natural numbers would be possible for all dividers of 72

divider	(144)	72	36	24	18	12	9	8	6	4	3	2	1
number	5	10	20	30	40	60	80	90	120	180	240	360	720

These solutions, however, are certainly not obvious. They cannot be seen intuitively and would only be accessible to the pupils after they had been formalized further. They all refer to the function

$$\text{number of sections} \cdot \text{common}\begin{Bmatrix}\text{divider}\\ \text{multiple}\end{Bmatrix} = 720$$

and the next step would bring all natural number solutions (x, y) of the equation $y \cdot x = 720$ with no correspondence to the factual situation.



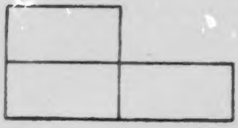
solutions of the children

THE AREA CALCULATION AND THE FUNCTION $y \cdot x = 20\,736$

The second aspect, too, offers a great variety of solutions. It is apparent that the pupils who were allowed to work without an interference from their teacher, show no uniform approach. Principally, there are two different solution strategies. With one of these solution strategies, the total area is calculated first, then a third of this number. Apart from the standard solution

$$(\text{length} \cdot \text{width}) : 3$$

there was a plethora of solutions, which all proceeded from the possibility of area transformation.



The total area remains unchanged if (for example) the width is doubled and the length is halved (in return). This would result in a square. It is typical for the behavior of the pupils that they are not content once they have found a solution, but are always eager to find new solutions. This attitude of the pupils is remarkable in a number of ways:

1) In each case they have the possibility of controlling themselves (since they have long known how large the area is). Thus the pupils obtain a direct feedback from the factual situation as to whether their solution is "right" or "wrong".
2) The pupils obviously enjoyed finding ever new solutions.
3) In so doing, the search for new possibilities becomes an end unto itself. The distance from the original problem becomes bigger and bigger, and the known solution - through reversing the problem - becomes the originating point and the key to the solution. For example, if the pupil wants to solve the following task:

   The new width is to be 9 meters

   He uses the total area (20 736 squ.m.) and arrives through division by 9 at the new length of 2 304 m.
4) The activities of the pupils have shifted emphasis: away from solving a simple practical task - towards investigating a mathematical context - irrespective of what the original problem might have been.
   Notably, this here is the function $y \cdot x = c$

The generalization process is being put into action through such questions as

- are there other solution possibilities?
- are there any more solution possibilities?
- how many solution possibilities do we find?
- how many solution possibilities are there?

With the method of area transformation ($y \cdot x = ky \cdot \frac{x}{k}$ or $y \cdot x = \frac{y}{k} \cdot kx$) the pupils constructed 29 product configurations. By so doing they got an idea of the interdependence of y and x and of the great variety of possible solutions.

GRAPHIC & ALGEBRAIC PRESENTATION OF FUNCTIONS - CAN THE STUDENT RELATE FROM ONE TO THE OTHER ?

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Exploring the graphs and algebraic presentations of functions is a good topic for the application of diverse working methods in the classroom. While high school students can apply their knowledge of calculus, junior high school students have no calculus techniques at their disposal. This situation can lead to look for new methods in teaching graphs of functions, methods which stress discovery and may help develop fruitful thinking of students. The conventional method, which is calculating the coordinates of points, is inefficient and can sometimes lead to misconceptions in understanding the graphic presentation of functions and is sometimes prohibitive. A suggested approach to overcome the difficulties by applying "elementary methods" (with no calculus), was presented in PME 1981* in connection with the "quadratic function". The students were encouraged to explore the algebraic form of the quadratic function, get enough information about the "behaviour of the graph" to be able to construct it.

Students usually do not tend by themselves to relate the graph of a function to its algebraic presentation. We shall demonstrate an approach of exploration, which is not common in algebra but is found in geometry when solving certain problems. One can look upon a given geometric concept from different points of view. A segment can be a diagonal in a parallelogram, a side of a triangle or a segment that cuts through parallel lines, (transversal).

We shall here demonstrate how to bring out the different aspects of presentation of functions and relate from one to another. Two examples will be presented (they do not form an algorithm). The examples deal with graph exploration and can be solved by good 9th graders and on, assuming they had proper preparation *(Such as learning the quadratic function as demonstrated in the PME Proceedings 1981, in the paper of R. Hershkowitz and M. Bruckheimer mentioned above.)* Such examples are meant to influence the student's ways of thought, approach and flexibility in search of information by unconventional methods.

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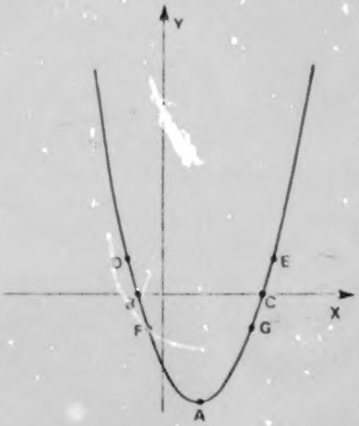
* R. Hershkowitz & M. Bruckheimer

The problems used as examples were chosen from a collection of suggested problems for teaching the graph of a function. Some were tried successfully with students and some with teachers.

Example 1 :

Given the algebraic form of the function $y = (x^2 - x - 2)^2$ the students are required to construct the graph. They can approach the problem only because they are familiar with the graph of $y = x^2 - x - 2$. They can construct the graph by squaring the value of y of each given point on the graph (see figure.) The squared graph is non-negative in the whole domain. To construct the graph we start from the given points on the graph.



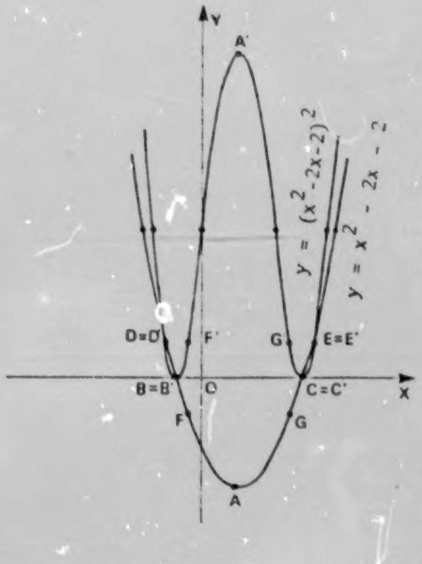
i) Point A is the vertex of the parabola and its minimum will be A', the maximum of the required function.

ii) Additional points that will help to construct the required graph are the fixed points, where $y = 1$ or $y = 0$ (as $1^2 = 1$ and $0^2 = 0$). These are the points B, C, D and E.

iii) Additional points are those with $y = -1$ which are G and F and their images G' and F' which are a reflection of G and F in x .

In the domain left of B and to the right of C the student can be assisted by the reasoning that $y^2 < y$ when $y < 1$ and $y^2 > y$ when $y > 1$.

The required graph will be below the parabola from B to D and above the parabola after D. Similarly we can get the graph to the right of C.



Example 2 :

Given the function $y = \frac{x}{x^2 + 1}$ Construct the graph.

We shall start out from the graphs of the functions:

$y = x$

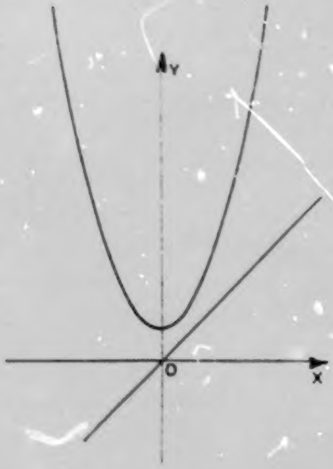
$y = x^2 + 1$

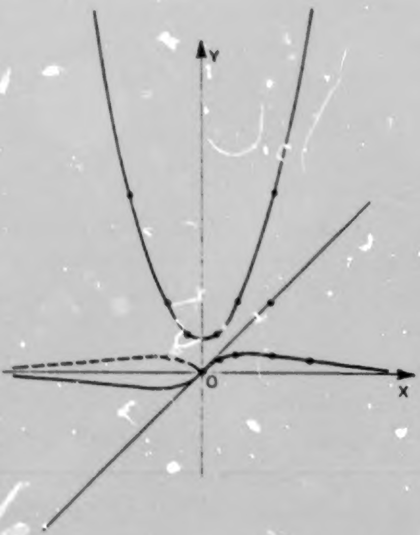
As $x^2 + 1 > 0$ for every x, the above given quotient is defined for all R. The quotient will be zero for $x = 0$ and hence the point (0, 0) is on the wanted graph.

To the right of zero $x > 0$ and therefore the quotient is positive. It is quite obvious from the figure that if we move "enough" to the right, the values $x^2 + 1$ will grow "much faster" than $y = x$ and the quotient will grow smaller and approach zero.

To the left of zero the quotient is negative, but the absolute value is symmetric to the right part.

In order to identify the point with largest y (or smallest), the following reasoning is used: $x^2 < x$ when $x < 1$ and $x^2 > x$ when $x > 1$. In the domain between 0 and 1 (or -1 and 0) we can see that $y = x^2 + 1$ will grow "slower" than $x = y$ and therefore the quotient will grow. From the point (1, 1) and on (or (-1, -1)) the process will be the opposite and the quotient will decrease. Hence, we shall have a maximum for $x = 1$ or minimum for $x = -1$.





Our starting point in the examples above was from the algebraic presentation of the function to its graph. It is of course interesting to see the opposite direction. This brought us to an 11th grade, of mathematics oriented students. The function explored was the parabola. The obvious thing one observes is that unless treated properly, the students usually do not do what we expected of them. Treatment is a necessity!

20 students participated in the experiment. The given function was

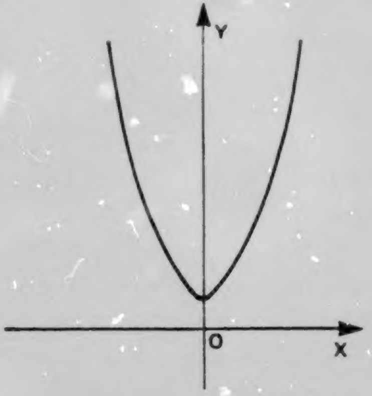
$$f(x) = \begin{cases} 2^x & x > 0 \\ \left(\frac{1}{2}\right)^x & x < 0 \end{cases}$$

and the students were expected to give answers to the following:

a) construct the graph of the function
b) is the graph a parabola? explain!

We would like to point out that the function f(x) resembles a parabola (see Figure)

5 out of the 20 students gave the right answer and explanation.
14 gave the wrong answer, stating that the graph was a parabola.
1 did not answer.



It is interesting to follow the explanations given by the students. The starting point of those who gave the correct answer was choosing pairs of numbers that belong to the algebraic formula given above, and proving algebraically that they cannot belong to any quadratic function.

The students that presented wrong answers claimed one of the following reasonings

a) The symmetry of the given function (6 students).
b) Symmetry and the exponent without connection of the place of the unknown and the value of the exponent (3 students).
c) Some properties of the function similar to those of the quadratic function, such as symmetry, minimum, decline to minimum and increase after the minimum (5 students).

It is interesting to point out that 3 students who knew and mentioned that $y = 2^x$ and $y = (\frac{1}{2})^x$ are exponential functions said the graph was a parabola.

In this work we pointed to a way to help develop the thinking of students in relating the algebraic presentation of a function to its graphic presentation. The importance of this topic is mainly because of 3 reasons:

a) Presenting a method independent of calculus in treating functions and hence suitable for younger students.

b) Educating students to be flexible in using different presentations of a mathematical concept.

c) The "function" is a basic concept in mathematics and is a part of modern education for several years. This is why using this concept to broaden the angle of approach to problems treating functions is of special importance.

THE NOTION OF PROOF — SOME ASPECTS OF STUDENTS' VIEWS AT THE SENIOR HIGH LEVEL

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§ 1. The Problem and the Method

In Fischbein and Kedem, 1982, it was claimed that the majority of high school students do not understand that "a formal proof of a mathematical statement confers on it the attribute of a priori, universal validity and thus excludes the need for any further checks" (p. 128). In this paper we will deal with a different aspect of a mathematical proof related to the following question: given a sequence of mathematical arguments, what makes it a mathematical proof in the eyes of the students? Two classroom impressions preceded this inquiry: (1) A computation is not considered a proof. (2) Many students when asked to prove a certain mathematical statement about a particular case repeat the general proof in terms of the particular case. This, instead of applying the general statement (which was previously proved) to the particular case.

In order to answer the above question we presented to our students an algebraic statement and its proof, then told them a "little story" and finally we asked them two questions about the "story". The algebraic statement and its proof were taken from Fischbein's questionnaire, 1982. The "little story" and the two questions which followed it were ours. Here is our full questionnaire:

Part A: In an algebra class the teacher proved that every whole number of the form n^3-n is divisible by 6. The proof was the following: $n^3-n = n(n^2-1)$. Using the formula $a^2-b^2 = (a+b)(a-b)$ we can write: $n^2-1 = n^2-1^2 = (n+1)(n-1)$. Thus $n^3 = n(n+1)(n-1)$. But $(n-1)n(n+1)$ is a product of three consecutive whole numbers. Therefore, one of them should be divisible by 2, one of them (not necessarily a different one) should be divisible by 3. Thus, their product should be divisible by $2\cdot3$, namely, by 6.

(a) I understand all the details of the proof and the proof seems to me correct.

(b) There are some details in the proof that I did not understand. They are the following:

Part B: A day after that the following exercise was given in a homework assignment: Prove that 59^3-59 is divisible by 6. Here are three answers which were given by three students:

1. I computed 59^3-59 and found out that it is equal to 205,320. I divided it by 6 and I got 34,220 (the remainder was zero). Hence the number is divisible by 6. Also, when using divisibility criteria we see immediately that the number is divisible both by 2 and by 3. Therefore it is also divisible by 6.

2. One can write $59^3-59 = 59(59^2-1)$. But $59^2-1 = 59^2-1^2 = (59+1)(59-1)$ (according to a well known formula). Thus: $59^3-59 = 59(59+1)(59-1)$. We've got a number which is a product of three consecutive numbers. One of them is divisible by 2 (58 in this case) and one of them is divisible by 3 (60 in this case). Therefore, their product is divisible both by 2 and by 3, thus also by $2\cdot3$, namely by 6.

3. In a previous lesson we learned that every whole number of the form n^3-n is divisible by 6, thus 59^3-59 is divisible by 6.

I. Write down which answer you prefer and explain why.

II. If there is no answer that you prefer or there are two answers that seem to you the same please specify and explain.

§ 2. A Mathematical Analysis

From the mathematician's point of view all the answers in the questionnaire are correct. Each of them is a proof of the statement that 59^3-59 is divisible by 6. Answer 1 is a direct proof — in this case a computation or a check. In answer 3 the general statement previously proved is used and therefore, at this particular context, it is the one which should be preferred. Answer 2, although correct looks silly at this particular context. If the general statement is proved it is totally unnecessary to repeat it in terms of the particular number 59^3-59.

Moreover, in our particular context also answer 1 would not be considered as a good answer and most mathematicians and mathematics teachers will prefer answer 3 to the other two answers. This was our impression from conversations we had with some mathematicians and mathematics teachers.

§ 3. The Sample and the Results

The questionnaire was administered to 365 high school students in the tenth and the eleventh grades. Part of the students were in high level mathematics courses and part of them were in low level mathematics courses. In the rest of the students, differentiation had not taken place yet. Thus, in our table we will relate to 5 groups: Group 1 — the whole sample (N = 365). Group 2 — the low level mathematics students (N = 109). Group 3 — the high level mathematics students (N = 87). Group 4 — the 10-th graders (N = 227). Group 5 — the 11-th graders (N = 138).

But before reporting the results of these 5 groups we would like to relate to two additional subgroups: students who claimed they understood the given proof — Group (a) (N = 310), and students who did not understand at least one detail of the proof — Group (b) (N = 55).

Table 1 — Distribution of Preferences in Groups (a) and (b)

	Answer 1	Answer 2	Answer 3	No preference
Group (a) (N = 310)	9%	37%	46%	7%
Group (b) (N = 55)	40%	24%	25%	11%

The difference is significant at 0.005 level and it is not surprising. If, at all, something is surprising it is the fact that of those students who did not understand the proof (at least to a certain extent) only 40% preferred answer 1 to the other two answers.

In their explanations some students expressed rejection to certain answers. This is shown in Table 2.

Table 2 — Distribution of Rejections in Group (a) and (b)

	Answer 1	Answer 2	Answer 3	No rejection expressed
Group (a) (N = 310)	15%	10%	9%	66%
Group (b) (N = 55)	7%	4%	5%	84%

Since half of the cells in Table 2 were too small we avoided a significance test. Also, note that students were not asked to reject answers, they were asked only to express their preferences. In spite of these, one can see a clear tendency in the table, a tendency which is consistent with the impression mentioned in § 1, namely, a computation or a check are not a proof.

Table 3 — Distribution of Preferences in Groups 1-5

	Answer 1	Answer 2	Answer 3	No preference
Group 1 (N = 365)	14%	35%	43%	8%
Group 2 (N = 109)	18%	32%	35%	15%
Group 3 (N = 87)	9%	39%	52%	0%
Group 4 (N = 227)	14%	34%	44%	8%
Group 5 (N = 138)	14%	41%	41%	5%

The only significant difference here (at the level of 0.005) is between low level mathematics students and high level mathematics students. Age does not make a difference.

§ 4. Categories of Reasons and their Distribution

The fact that a student prefers a certain answer is, of course, not the only fact that counts. We are interested also in the reasons for his preference. Hence, we classified the reasons students gave for their preferences. For each answer we found several categories of reasons for preferring it. The k-th catebory of reasons for preferring the i-th answer will be denoted by category i, k. If no reason is given by the student or the reason is not classifiable we will denote it by (*).

Category 1,1: The answer is simple, short, clear and natural.

Category 1,2: The answer does not rely on previous proof. It does not include formulae.

Category 1,3: The answer confirms the formula. It shows that the formula is correct.

Since no significant difference was found between the subgroups of groups 1-5 who preferred answers 1, 2 or 3 we will bring only the distributions in the entire sample (N = 365).

Table 4 — Distribution of Reasons for Preferring Answer 1

	Category 1,1	Category 1,2	Category 1,3	(*)
(N = 51)	51%	29%	6%	7%

Category 2,1: The substitution confirms the general formula; it makes it concrete; it explains the general proof. Answer 2 is a reconstruction of the general proof; the student who gave it showed understanding of the general formula. By this method one can easily solve many similar exercises. It shows how the student got the answer. It contains the method of the proof. It has steps. This was an exercise and in exercises it is desirable to repeat the general procedure.

Category 2,2: The answer is general, clear, simple and short.

Table 5 — Distribution of Reasons for Preferring Answer 2

	Category 2,1	Category 2,2	(*)
(N = 127)	65%	28%	8%

Note that 28% of the students who preferred answer 2 to the other answers justified it by irrelevant reasons, since how can one claim that answer 2 is general, clear, simple or short?

Category 3,1: The answer relies on a formula which has been proved. It relies on a general proof. One can substitute letters for numbers. It is enough to prove a mathematical formula once, in a general way. It is safe because there is a general formula in which there are no mistakes (like in computations).

Category 3,2: The answer is simple, clear, short, less complicated than other answers and easy to remember.

Table 6 — Distribution of Reasons for Preferring Answer 3

	Category 3,1	Category 3,2	(*)
(N = 158)	76%	13%	11%

§ 5. Discussion

From a mathematical point of view the surprising part of the results was that a relatively high percentage of the students, in the context of the given questionnaire preferred answer 2 to the other answers.

It certainly shows that at least one third of the students in our sample (academic high school students) does not understand the nature of mathematical proof. (This does not mean that the other two thirds do understand it. It only means that on the ground of their reaction to this questionnaire it cannot be claimed that they do not understand it.) The question is whether, on the ground of this questionnaire, we can say something about the way that at least this one third of the students does view mathematical proofs. We must say immediately that any suggestion we will make will have a speculative nature. But when implicit views are considered, speculation is the only way (of course, any speculation should be examined later on against new and old data). It seems to us that category 2,1 is more than a hint to the following interpretation:

(1) The general proof is a method to examine and to prove a particular case (this view is deeply related to the understanding of variables and substitution in mathematics which requires a special research as well).

(2) A mathematical proof has to have a certain form; it has certain features. To verify and to prove are not necessarily the same. A verification is not necessarily a proof. Thus, a proof has a certain ceremonial aspects.

It is clear that this interpretation needs more evidence to confirm it. At this stage we can only point out that it is consistent with Fischbein's findings (1982). Even the percentages are close. Between 16% to 31% of the students in Fischbein's sample are "consistently empirical" (namely, they systematically do not think that "a formal proof can guarantee the general validity of a statement" (p. 130)). This thought can be the reason or the result of the views expressed in category 2,1, shareable by 23% of the students in our sample.

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LINGUISTIC BARRIERS TO STUDENTS' UNDERSTANDING OF DEFINITIONS

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A study was conducted of students' understanding of defined concepts in a first course involving deductive mathematics. At UC, Berkeley, there is a sophomore course on linear algebra where, for the first time, students must grapple seriously with definitions. In contrast to computational courses, few explicit procedures for problem-solving are provided, and these apply only to a small portion of the exercises. In most cases, students are called upon to write proofs based on formal definitions, axioms, and other theoretical statements. The procedural information requisite for theorem-proving is to be extracted from these formal statements, which students encounter routinely, both on the classroom blackboard and in the course textbook. Students, however, typically arrive in such courses altogether ignorant of many subtle conventions of mathematical writing and entirely inexperienced in the kinds of analyses necessary for extracting information adequately from mathematical text. Further difficulties are posed to beginning students by the need to use mathematical language actively in communicating their ideas. Indeed, a quantum leap in difficulty is experienced by most students at this point in the mathematics curriculum. The present study concentrated, in paticular, on students' difficulties in coping with the formal definitions by which concepts are introduced. These difficulties hamper students in contending with all aspects of the course, but they surface most clearly when exercises require the ability to construct proofs by "working literally from" definitions.

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OUTLINE OF RESEARCH METHODS

Analyses were performed in two broad areas for the purposes of this study. One major focus was on students' linguistic (mis)behaviors with defined terms. This aspect of the research utilized "ready-made" data, provided by students' spontaneous writing on regular homework assignments. Additional data were provided by students' written work on tasks especially designed to probe for the presence of certain aspects of understanding singled out for investigation, as well as by the written transcripts obtained from extensive interviews conducted with individual students. These interviews concentrated on students' performance of these tasks, on course concepts, and on passages of the course textbook. A second focus of the study was on the course exercises themselves and, in turn, on the textual passages from which the information requisite for the solution of these exercises is to be derived. Findings based on these coordinated undertakings are briefly summarized below.

MAJOR FINDINGS ABOUT STUDENTS

The written work submitted by students on regular homework assignments reveals the extent to which the writing of mathematical prose is a new and foreign activity to them. In the first place, students often fail to acquire correct usage of course terminology. This information is implicitly prescribed by definitions, but students apparently miss it. In addition, they are not yet facile with the introduction and handling of symbols, a basic activity in proof writing. Their errors in this activity result in writing which is at best ambiguous. Indeed, the writing of the overwhelming majority of students is pervaded by abuse of syntax, usage and grammar, and by various sorts of unclear expression. In particular, students employ linguistic constructs which are syntactically inappropriate (and often ambiguous or meaningless) for referring to the objects investigated in the course. Thus, for example, their writing often fails to distinguish a set from its members. Also, it juxtaposes terms that do not belong together, because they denote objects of wrong categories. As communicators, then, students do not participate in many normal conventions of mathematical writing, and their utterances are abundantly flawed. Based on their prose, it is often very difficult to assess to what extent these students are communicating good ideas poorly, and to what extent, and in what ways, the linguistic abuse is indicative of poor understanding of the concepts on their part. Subsequent interviews with the authors of this garbled prose often helped to identify cases in which students' mismanagement of language can be linked to interference with understanding of course concepts or to flawed apprehension of the mathematical entities investigated in the course.

As various deficiencies were identified in students' spontaneous writing, further, more systematic documentation of their behaviors was sought. Such data were elicited by the study's test items. One such task revealed that late into the term, students cannot adequately articulate the precise denotation of a standard course symbol such as "R^5", used consistently throughout the course for referring to a very specific vector space. Most students failed to indicate precisely what kind of object is denoted by the particular symbol, or did so incorrectly. Among those who indicated that a certain set is involved, there was failure to specify its precise membership. In fact, in one class, only 2 of 19 responses elicited were deemed to be adequate both in content and in language, while 3 of the 19 responses failed totally to suggest the correct denotation of the symbol.

These and other data suggest that these students have no clear, consistent, unambiguous meaning associated with a symbol such as "R^5" as an entity unto itself, encountered outside an immediate problem-solving context. This inability to articulate the denotation of standard course symbols, even late into the term,is striking, as is students' own idiosyncratic use of these symbols. Indeed, they seem to lack the idea, basic in mathematical writing (and elsewhere, of course) that symbols acquire a meaning through conventions agreed upon by a community.

TEXTBOOK TREATMENTS AND INSTRUCTIONAL PRACTICES RELATED TO STUDENTS' DIFFICULTIES

In analyzing students' flawed utterances, the researcher bore in mind that this linear algebra course is a natural point in the mathematics curriculum where students have to increase sharply their use of mathematical language; it is to be expected, then, that their communication at this stage will be flawed and awkward. At the same time, a detailed examination of certain features of written mathematical communication, and of the kinds of exercises which students are routinely assigned, helps to shed light on students' difficulties.

The difficulties cited above with respect to the course's standard symbols, for example, can be linked to the fact that some of the definitions which introduce them are actually "buried" in the course textbook, at times in the context of exemplifying some newly-introduced concept. Thus, the symbols "R^n", "M_{mn}", and "P_n" -- denoting certain specific vector spaces to which reference is frequently made in the book's subsequent discussion -- are introduced in the text seemingly in passing, each under the caption, "Example" (of a vector space). It is not explicitly made clear that each of the symbols introduced in these examples is to retain the same denotation for the remainder of the book. (The subtle cues to this effect, which are picked up by the skilled reader, are not noticed by students.)

The same textbook buries its definition of the symbol "0", as it is used in some contexts for denoting a matrix all of whose entries are the number zero, within a proposition concerning matrices: "Proposition $A + (-A) = -A + A = 0$, where 0 denotes the matrix of the same order as A, all of whose entries are 0." One difficulty with this passage is that the same symbol "0" refers both to a number and to a matrix. Numerous other instances were found which bring to light the complexities entailed in clearly associating a symbol with its correct referent, as intended by a mathematics author. A further, more significant difficulty illustrated in the passage cited above is its implicit universal quantifier, and the implicit range of the variable which it employs. The proposition may be reformulated as follows, to bring out more explicitly its intended meaning: "Let A be any nxm matrix. Then each of the matrices A + (-A) and -A + A equals the nxm matrix all of whose entries are 0." These interpretations must be assumed in order tomake sense of the statement of the proposition. Similar kinds of interpretations are needed in order to comprehend the intended meanings of exercises. One such example is the following. Suppose that A is a matrix for which $A\underline{x} = \underline{0}$ for all $\underline{x}$. Show that A = 0. Many students are totally puzzled as to the intended meaning of this task. They are not yet skilled in furnishing for themselves the information which is only implicitly conveyed by the text.

Indeed, mathematical writing is greatly abbreviated, a practice which certainly makes communication among "insiders" more efficient; but it becomes a stumbling block to communication when some of the participants are uninformed about the conventions employed. Such is the case at hand. The students are found to be uninformed, and the instructors uninformative, perhaps because they are unaware of the large gap in linguistic conventions which separates them from their students. Nothing in the instructional design explicitly addresses the kinds of difficulties to which this study points, so that certain understandings related to mathematical language and concepts are acquired with difficulty, if at all, as a by-product of students' routine participation in the course.

SUMMARY

The research reported here points to a significant gap in the communication system which prevails between students and their instructors in a first deductive course. As communicators, students do not participate in many normal conventions of mathematical writing, and the interpretation of their flawed utterances is frequently left to the imagination of the sympathetic instructor. As recipients of communication, they do not extract from mathematical text and from the formal statements of definitions, in particular, that information which is intended by the author. Moreover, this study's attempts to locate sources of these various deficiencies makes clear that students do not share with their instructors various understandings concerning the interrelation of mathematical objects, symbols, and definitions. They are particularly hampered thereby in contending with the assigned theorem-proving.

The problems described here are of significant import to pedagogy, perhaps at earlier points in the mathematics curriculum as well. They need to be widely recognized, both by instructors and by the authors of mathematics textbooks which are addressed to beginning students.

MATERIALS TO BE MADE AVAILABLE AT THE CONFERENCE

--- samples of students' written work, exemplifying categories of linguistic misbehavior

--- sample interview transcripts

--- recommendations for addressing problems outlined here

RATIONAL NUMBERS AND DECIMALS AT THE SENIOR HIGH LEVEL — DENSITY AND COMPARISON

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§ 1. Introduction

This paper is a part of a broader study which examines some students' views about real numbers at the senior high level (ages 15-17). A questionnaire was compiled and administered and the results concerning two of its questions are brought and analysed here. All the students of our sample studied at the same high school, which is academically selective. Hence, our results will not reflect the general level of mathematics students in the entire country. It is probably lower, but we cannot say how much. Nevertheless, the results will reflect the mathematical level of students in academically selective high schools.

§ 2. The Density Aspect of the Rational Numbers

This aspect is examined in K.M. Hart, 1981, in *two specific* cases with age groups 12-15 (students were asked to write down a fraction that comes between ½ and 2/3 (p. 73) and to tell how many different numbers there are between 0.41 and 0.42 (p. 55)). The success percentages for all age groups were 36 and 12, respectively). We examine this aspect by the following *general* question:

Students were told that in an algebra lesson on rational numbers a teacher had written down two rational numbers and had asked his students whether there were more rational numbers between them and if they were — how many. One of the students to whom this was told claimed that the answer depends on the two given numbers. A second student claimed that there are always numbers between two given rationals but the number of these numbers depends on the two given numbers. A third student claimed that there are always numbers between two given rationals and the number of these numbers does not depend on the given numbers. Who is right and why?

Four categories of answers were found.

Category I: Between two given rationals there is a finite number (sometimes zero) of rationals which depends on the two given numbers.

Category II: Between two given rationals there is always another rational. This is said without relating to the question how many. Thus, it is impossible to tell whether the student knows that there are infinitely many such numbers. (In the context of the questionnaire, such an answer is considered by us weak since the students had enough stimulus to say all they knew about the question.)

Category III: Between two given rationals there is an infinite number of rationals which depends on the two given numbers.

Category IV: Between two given rationals there is an infinite number of rationals which does not depend on the two given numbers.

The answer distribution is given in Table 1.

Table 1 — Answer Distribution

	Category I	Category II	Category III	Category IV	No answer
Grade 10 N = 91	18%	19%	2%	59%	2%
Grade 11 N = 85	22%	11%	0%	61%	6%

Here are also some answer examples which demonstrate the students' thoughts.

Category I: 1. *I think the first student is right because if the two numbers are consecutive then it is impossible to write down numbers which lie between them. For instance ½ and 2/3.*

2. *The second student is right because it is impossible to claim that there are infinitely many numbers between two given numbers. The amount of such numbers depends on the given numbers and even if there are many of them they have an end.*

Category II: *For any two rationals, even very close to each other, it is possible to find numbers between them. For instance, 1/29 and 1/30. There are numbers between them like 59/1740.*

Category III: 1. *Between any two rationals there are infinitely* *many rationals (for instance the mean of the two rationals). But in spite of that, there is a difference between the infinity levels.*

2. . . . *between 5 and 10 there are more numbers than between 6 and 7.*

(Note that the view expressed in this category is typical to views about infinity reported in Fischbein *et al.*, 1979.)

Category IV: *It is always possible to divide by 2 the difference between two rationals and to add the result to the smaller number. The result is again a rational number. One can go on and divide by 2 the difference between the smaller number and the new number. The sum and the difference of rationals are always rational.*

In addition to the above examples we would like to draw the reader's attention to some additional points which were found in the analysis and are worthwhile to mention.

(a) The role of the irrationals in arguments about density.

Most students did not mention at all the irrational numbers. From the mathematical point of view they are irrelevant. However, a few students related to them in spite of that. This was done in a way that indicated a beginning of an accommodation process; namely, quite often the ideas were wrong but some ties between the concept of the irrationals and the concept of the rationals were formed. For instance: 1. *The student who claimed that the answer depends on the numbers is right, since it is possible that the two given numbers are consecutive and there are no rational numbers between them; for example: two numbers between which lies the number π which is irrational.*

2. *Between ½ and 1/3 there are no rational numbers because 1/2.5 is not rational since it is not presented as an integer over an integer.*

3. *The third student is right since between any two numbers there are infinitely many numbers and therefore probably some of them are rational.*

(b) Abstract thinking versus concrete thinking.

This is a general issue not necessarily related to our particular topic. Some students carry out their mathematical arguments in terms of specific numbers while others are capable of carrying them out in terms of semantic variables; namely, they use general arguments. This is beautifully demonstrated by the answers representing categories II and IV above.

(c) The role of the number line in students' thinking about the density aspect.

Several students established their answers on number-line arguments. This is interesting because it happened spontaneously and thus shows that the number line is spontaneously used to help students' thinking about numbers. Here are two examples:

1. *The distance on the number line determines the number of numbers between two given numbers; if the distance is greater there are more numbers.*

2. *Always between two points one can denote another point and I assume that this law is true also about rational numbers.*

§ 3. The Comparison Between Decimals

The students' ability to compare decimals was examined by the following question:

Here are some decimals. Write down all the numbers which are equal to each other and arrange all the different numbers in an increasing order.

(a) 0.33333; $0.3\dot{3}$; $0.29\dot{8}$; 0.4; 0.299; $0.33\dot{3}$.

(b) 0.999; $0.\dot{9}$; 2; $1.00\dot{1}$; 1; 1.0000001.

Please, explain your answers. (The notation $0.3\dot{3}$, etc. was taught to the students and also re-explained to them in the questionnaire.)

We will discuss first two strategies of decimal comparison explicitly expressed in students' answers.

Strategy I (explicitly expressed by 33% of the 10th graders (N = 91) and by 13% of the 11th graders (N = 85)). It is based on the belief that the more digits a number has after the decimal point the smaller is the number. Thus: $0.33\dot{3} < 0.33333$, $0.\dot{9} < 0.999$, $1.00\dot{1} < 1.0000001$, $1.00\dot{1} < 1$. (This was surprising but the fact that students were systematic together with their written explanations rule out the interpretation that students just ignored the upper dot denoting the decimal period and thus by writing $0.\dot{9} < 0.999$ they really meant $0.9 < 0.999$.) An answer demonstrating this strategy is the following: *As much as the decimal has more numbers after the decimal point — it is smaller since the denominator is increasing.*

This phenomena can be explained by the inability to simultaneously relate to more than one factor involved in the situation. Piaget claimed it about children in the beginning of the concrete operation stage, but perhaps this inability is not a question of age and it is typical to relatively complicated situations with a factor that seems to be dominant (like the height in the Piagetian experiment and like the claim that the greater is the denominator the smaller is the number which is, of course, true as long as the numerator remains the same).

Strategy II (explicitly expressed by 19% of the 10th graders and by 34% of the 11th graders): One should compare the decimals digit by digit (this is of course a correct strategy in case the two decimals compared are both infinite or both finite. It does not always work in case one decimal is finite and the second one is infinite. Thus, some students who used it successfully with $0.29\dot{8}$ and 0.299 failed in the comparison of $0.\dot{9}$ and 1 claiming that $0.\dot{9} < 1$ (see also Tall, 1976)). A typical description of strategy II is the following: *I go through the digits from left to right. The first digit which is different, if it is greater in the number A than in the number B then A > B.*

It is worthwhile to mention that more than half of the students did not describe their strategies for decimal comparison.

In addition to the above strategies also two interesting perceptions of the infinite decimal were expressed in the students' answers.

Perception I (explicityly expressed by 20% of the 10th graders and 18% of the 11th graders): The infinite decimal is perceived as one of its finite approximations.

Thus about 17% of the students suggested that $0.3333 = 0.33\dot{3} = 0.3\dot{3}$ or $0.\dot{9} = 0.999$. This was justified by arguments like the following:

1. *The difference is so small that it is hardly noticed. Therefore, one can say that the things are equal.*

2. *It is possible to write $0.999 = 0.\dot{9}$ since three digits after the decimal point are sufficient, otherwise it is not practical.*

Perception II: The infinite decimal is perceived as a process, not as a product. This can be also characterized by the term: the dynamic perception of the infinite decimal.

In the answers expressing this perception the infinite decimal is considered as something which grows and has tendencies.

1. *An infinite decimal increases a little bit all the time.*
2. *$0.\dot{9} < 1$ because it will never reach 1.*
3. *$0.29\dot{8}$ is smaller than 0.299 because all the time it wants to reach the finite number 0.299 which is also the closest number to $0.29\dot{8}$.*

Note that the last perception can be considered as a starting point for a correct mathematical concept. Mathematicians associate tendencies to sequences, not to single numbers. However, there are students who have the feeling that a single infinite decimal itself tends to something. This is, perhaps, a first step toward the concept of the infinite decimal as a limit of an infinite sequence of finite decimals.

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A DIAGNOSTIC TEACHING PROGRAMME IN ELEMENTARY ALGEBRA: RESULTS AND IMPLICATIONS

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This paper presents an overview of the work done in elementary algebra (generalised arithmetic) by the Strategies and Errors in Secondary Mathematics project, a project funded by the Social Science Research Council and based at Chelsea College from 1980 to 1982. This project aimed to investigate the causes of certain commonly made errors in elementary algebra, and to use this information in order to develop a teaching programme aimed at helping children to avoid making these errors.

ANALYSIS OF ERRORS

By conducting a series of individual interviews with children identified as making particular errors in elementary algebra, the project was able to describe several areas of difficulty which appeared to underlie these errors. These areas of difficulty have been described more fully elsewhere (e.g. Booth, 1982, 1983a, b) and are summarised as follows:

(1) Interpretation of letters. Children often (a) confuse letters as representing a number with letters as representing an object (Küchemann, 1980) and (b) think a letter always represents a single unknown value (Collis, 1975).

(2) Notation and convention. Some errors are due to misconceptions concerning algebraic notation such as (a) conjoining in algebraic addition, (Davies, 1978; Matz, 1980), ($a+b \rightarrow ab$), (b) not appreciating the need for brackets (see also Kieran, 1979).

(3) Formalization and symbolisation of method. Children often (i) do not make explicit the procedures used in arithmetic, and (ii) in fact often use informal or 'invented' (cf. Ginsburg, 1977) procedures which are not readily represented in an arithmetic (and hence algebraic) manner. Even when an appropriate formal method is used, children (iii) may not be able to symbolise it correctly, or (iv) may not realise that such symbolisation is what is required.

These areas of difficulty can be accounted for in terms of the Piagetian differentiation between concrete and formal operational thinking (see Collis, 1975), or in terms of the child's continuing to work (innappropriately) within an arithmetic 'framework of reference' (see Matz, 1980).

THE TEACHING EXPERIMENT

Using this information as a basis, a short teaching programme aimed at structuring children's thinking so as to enable them to avoid making the errors in question was developed by means of a series of small-group teaching experiments.

This teaching programme was based upon the need, identified from the interviews, to help children to:

(1) obtain a conception of letters as generalised number, i.e. representing a range of values;
(2) focus on the formal method required to solve a given problem;
(3) symbolise such methods correctly and unambiguously;
(4) appreciate that unclosed statements such as e+2 can represent 'answers';
(5) discriminate between expressions of the kind 'e+2' and 'ex2' in terms of the abbreviated symbol '2e';
(6) appreciate the need to use brackets.

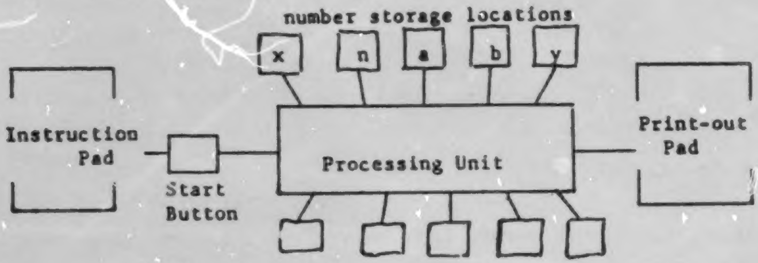

Fig.1

A teaching framework appropriate to these aims was provided by setting the instruction within the context of a 'mathematics machine' (Figure 1) which was to be 'programmed' to solve given classes of problem. The teaching thus emphasised both the need to analyse problems in terms of their formal structure, and to represent this structure in a concise and unambiguous manner, using the appropriate mathematical notation, and using letters as generalised number'.

Since it was hoped that the teaching sequence would be useful both in the initial teaching of algebra, and for remedial purposes, the effectiveness of the programme was investigated with groups of children who had not yet learned algebra, as well as with older groups containing children who were known to be making the relevant errors. The teaching programme covered a period of six to seven, 35 minute lessons, and was taught by:

(a) the researcher, using one group of 'novices' in algebra (aged 12) and three groups of children aged 13, 14 and 15 years respectively, who had been identified as making the errors in question.

(b) volunteer class teachers, using one group of 14 year old 'novices', and 3 groups of 13 year olds, 2 groups of 14 year olds and 1 group of 15 year olds. These latter groups were whole classes and therefore included children who were not making the identified errors, as well as a proportion (usually 40 percent or more) who were making the errors under study.

In each case, parallel forms of an algebra test designed to asses the occurence of errors relating to the areas of difficulty described above, and taken from the CSMS Algebra test, were given as pretest, as immediate posttest following the teaching programme, and as delayed posttest administered two to four months later.

RESULTS OF THE TEACHING EXPERIMENT

In general, the teaching programme was effective in promoting an improved performance, in terms of total score (max = 21), and this was maintained on the delayed posttest. Average gain scores between pre- and delayed posttests were in the order of 8-12 for groups of children identified as making the errors, and 3-7 for 'mixed' whole classes, i.e. groups also containing children who were not making the errors under study.

Analysis of individual or groups of items particularly relevant to each of the areas of difficulty outlined above, indicated that improvement was obtained primarily in the following areas:

(1) Conjoining in algebraic addition. All groups made notable gains, regardless of age or level of mathematical ability (as indicated by position of class in year-group). (See Figure 2).

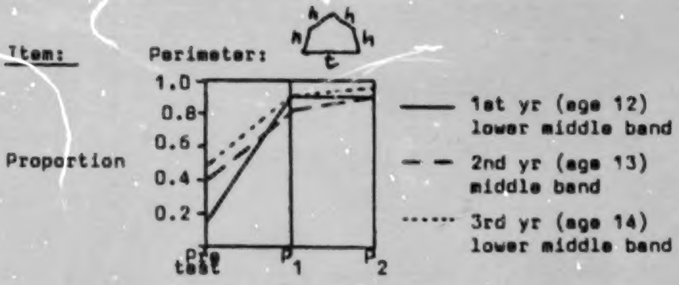

Fig.2 Proportion of correct answers on pretest, immediate posttest (P_1), and delayed posttest (P_2).

(2) Formalization of method. Notable gains were likewise made for all groups (see Figure 3a). One exception to this was noted in the case of the item represented in Figure 3b. However, since this item was symbolically more complex, involving as it may the use of brackets, it is suggested that the lack in improvement on this item was due more to problems of symbolisation than formalization. This is supported by the marked decrease in error (i.e. the giving of a numerical or algebraic 'code' answer) on this item, suggesting that children were moving towards a non-numerical representation of the problem but were as yet unable to symbolise it correctly. The nature of the test did not permit the separation of the formalization and symbolisation aspect of the problem.

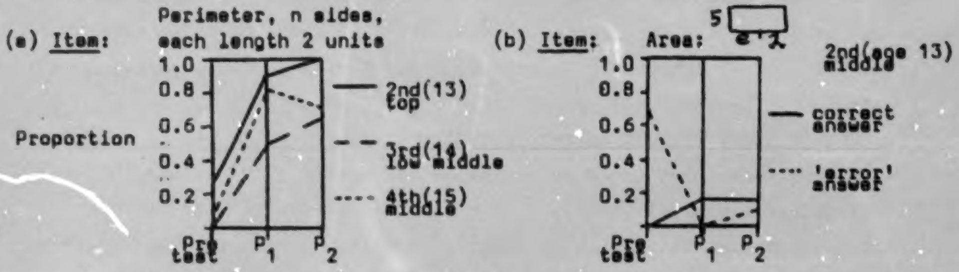

Fig.3 Proportion of correct answers (item (a)) and correct and 'error' answers (item (b)) on pre- and posttests.

Improvement with regard to the following area was observed only in the case of selected groups, namely the older children (15 years old) and younger 'top stream' children (aged 13 years) (see Figure 4).

Item: L+M+N = L+P+N True: always/never/sometimes, when....

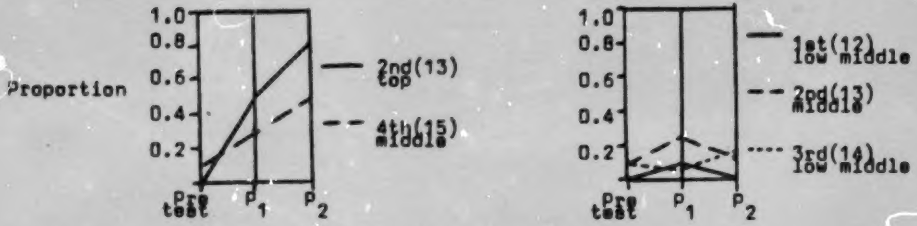

Fig.4 Proportion of correct answers on pre- and posttests: Comparison of groups.

(3) Letter as generalised number. Of particular interest in this respect was the observed improvement over time between the immediate and delayed posttests. An improvement in performance of this kind might be expected if children had not found the ideas embodied in the items readily assimilable, but needed time in which to construct an appropriate cognitive viewpoint with regard to the ideas concerned. The observation that only the older or mathematically more able children showed this improvement also suggests that there may be maturation-linked factors contributing to the child's likelihood of assimilating the notion of generalised number (see Figure 4 above).

Also of interest were the following observations:

(1) Many of the errors in elementary algebra investigated in the present research were also made by naive (in the sense of lacking previous exposure to algebra) pupils. The proportion of naive pupils making these errors was in many instances comparable to that observed for older 'more experienced' pupils.

(2) Children may respond correctly to items requiring the use of certain notation or convention (such as the use of brackets or the writing of an algebraic sum), and yet be unable to discriminate between correct and incorrect representation, i.e. be unable to select the correct and incorrect alternatives. This suggests that the issue of symbolisation is itself a complex one, and that understanding of notation may itself proceed via stages.

SUMMARY

The difference in effectiveness of the teaching programme with regard to the areas of difficulty identified suggests that these areas may not be of the same level of difficulty. This may mitigate against the employment of a strictly Piagetian interpretation of these difficulties. Particular problems do seem to be related to the notion of generalised number, however, and the findings indicate the possible involvement of a 'cognitive readiness' factor in accepting this idea. However, the observation that the (pretest) incidence of errors studied was of a similar magnitude in groups of children from first year (age 12) to fourth year (age 15) suggests that the fact of cognitive growth does not in itself ensure the growth of understanding in these areas. Attention must perhaps also be paid to the 'framework of reference' or 'knowledge' which the child constructs with respect to the topic in question. More research is needed in order to clarify this point.

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STUDENTS' MISCONCEPTIONS OF THE EQUIVALENCE RELATIONSHIP

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The equivalence relationship is one of the most fundamental concepts in mathematics. Yet, recent studies (e.g. Kieran, 1981) have pointed out that "the concept of equivalence is an elusive one not only for elementary school students but for high schoolers as well" (p. 324). It seems that children at various age levels interpret the symbol of equivalence, the equal sign, as an operation rather than a relation symbol. As a result, children at first and second grades have difficulties in reading arithmetic sentences not including operations (e.g. $5=5$), or not reflecting the order of operations (e.g. $\square = 2+3$) (Ginsburg , 1977). Children at sixth grade tend to change reflexive expressions to transitive ones involving mathematical operations (Behr et al., 1976). Even high school students still possess the thread of interpreting the equal sign as a "*do something signal*" (Italic in source) as indicated by errors they make in writing down expressions like the following:

$$a+b+c+d=a+b=x=x+c=y=y+d=e$$

In all these studies the equal sign was mistakenly viewed as a symbol which separates a problem and its answer with slight awareness for the notion of equivalence.

These findings raise the question as to the extent to which college students are able to overcome the misconceptions associated with the equivalence relationship. Do college students also interpret the equal sign as an operation symbol? Do students at this age level have difficulties in comprehending the three equivalence laws of Reflexive, Symmetric and Transitive? Do these misconceptions result from low achievement in mathematics, or from weak skills in solving equations? The purpose of the present study was to investigate these questions evaluating the underlying notions of equivalence among college students.

METHOD

Subjects

Participants were 150 college students majoring in education. They all passed the Matriculation Examination (Bagruth) in mathematics with an average score of 76. Approximately 80% of the subjects had studied high school courses in algebra, geometry and trigonometry, and the reminder (N=33) had taken also an introductory course in calculus.

Measurements

A fifteen-item algebra test was administered to all subjects at the beginning of their freshman year. The test was constructed as follows: eight multiple-choice items assessed misconceptions associated with the three equivalence laws, two open questions involved interpretation of the 'equal' and 'greater than' signs, and five linear equations dealt with one variable were to be solved. In one open question, for example, students were given a definition for mathematical operations and another for relations. A row of symbols was presented, and the students were asked to identify those representing operations and those representing relations.

In addition, students' scores in the High School Mathematical Matriculation Examination were analyzed.

RESULTS AND DISCUSSION

Interpretation of the Equal Sign

Only 56% of the students gave response indicating that the equal sign is relation symbol. However, students had less difficulty in interpreting the meaning of the symbol '<' than the symbol '='. While 19 % of the students indicated that '<' is an operational symbol, 44% of them viewed '=' as a "do something signal". This difference is statistically significant (chi-square =10.08; $p<.005$).

Why did the sign '=' mislead more students than the sign '<'? Clearly, in order to determine whether a is greater than b, students use a two-step procedure in which they first subtract the two elements and than judge according to the result obtained. The symbol '<' is used only after the second step when the subject has to present the relation between the elements. In contrast, the symbol '=' is usually used to obtain the answer, not to relate the elements presented in the problem. Thus, students tend to extend the set of mathematical operation symbols to include the equal sign but not the "greater than" sign.

Comprehension of Equivalence Laws

Most students (80%) were able to identify the identity of expressions presented in parallel forms (e.g. $2X+3=-5X+10$ and $-5X+10=2X+3$). However, many students (approximately 52%) have difficulties in identifying a solution set satisfying the equation $\underline{t+15=t}$ which was used to assess the understanding of the Reflexive Law. The common error was "any pair of numbers their difference equal 15". Probably, being unable to compare $\underline{t+15}$ and $\underline{t}$, students assigned different values to the two $\underline{t}$'s presented in the equation. This response closely resembles that of elementary school children who interpreted the equality $3=3$ as means $6-3=3$ or $7-4=3$ (Behr et al. 1976).

Many students (approximately 60%) also have difficulties in identifying identical expressions presented in transitive forms. For example, students were given two sets of equalities, $2a+3=5a+7$ and $5a+7=3a+1$ and were asked to judge whether $2a+3=3a+1$. Apparently, subjects could not assign any meaning to indeterminate forms such as '5a+7' and thus looked for a number which satisfies the given equalities.

Misconceptions associated with the equivalence relationship were manifested also in the translation of verbal expressions into algebraic sentences and vice versa. Only 30% of the subjects were able to translate the sentence "for each five tanks there are two airplanes" or to assign the correct meaning to the expression $\underline{5a=2t}$ where $\underline{a}$ presents the number of airplanes and $\underline{t}$ presents the number of tanks. In both cases students used a "word order matching strategy" reported also by Resnick and Clement (1980) in which students translate directly from the problem to an equation and vice versa without due regard for the semantics of the problems or for the mathematical meaning of the equal sign. The same difficulty was manifested also when students were asked to identify the larger variable in the equlity $3k=m$. Approximately 38% of the students mistakenly answered that $k>m$ probably because the misleading notion that three k is greater than one m. Evidently, if students would have consider the equal sign as a symbol of equivalence they would not make such mistakes.

Mathematical Achievement and Equivalence Misconceptions

In order to examine whether the equivalence misconceptions resulted from weak mathematical skills, students' ability to solve linear equations as well as their Matriculation Exam scores in mathematics were assessed in regard to their ability to comprehend the equivalence laws and the equal sign. Since more than 90% of the Ss succeeded in solving all the linear equations presented except the one involving fractions, it does not appear that their poor understanding of equivalence derived from a lack of skill or acquaintance with linear equations. Moreover, the Matriculation Exam scores in mathematics were only weakly correlated ($r= .04$) with success on the ten items assessing equivalence understanding.

SUMMARY

The results of this study show that some of the misconceptions of equivalence manifested in children remain throughout adulthood.

Eventually, many college students grasp the equivalence laws, but could not overcome the thread that the equal sign is a "do something signal". The findings that these misconceptions are weakly related to mathematics achievement raise the question of how do students possess these misconceptions, and more importantly, how could they be erradicated. This issue merits further consideration in future research.

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EXPERIMENTAL MODELS FOR RESOLVING PROBABILISTIC AMBIGUITIES.

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Some of the apparently most elementary probability problems can become quite intriguing when assumptions that have to be unequivocally spelled out remain unspecified. One way to expose such implicit assumptions, would be to actually perform the "experiment" that generated the problem. In this talk I wish to analyze a few puzzling problems in probability by offering experimental models that should clear all doubts concerning the interpretation of the problems.

Causal Reasoning versus Probabilistic Inference

The first problem requires no construction of an isomorphic "urn model", since it is an urn problem, first presented in the PME conference at Warwick (Falk, 1979): An urn contains two white balls and two black balls. We shake the urn thoroughly, and blindly draw two balls, one after the other, *without* replacement.

First we ask for the probability that the second ball is white, given that the first ball is white. Students have no difficulty inferring from the information that the first ball was white that $P(W_{II}|W_I)=1/3$; The main part of the problem comes now, when we ask for $P(W_I|W_{II})$. The modal answer, repeatedly given by a little more than half the students in introductory courses, is $P(W_I|W_{II})=1/2$. They reason as follows: Since the second draw cannot affect the first one, the knowledge of its outcome is irrelevant to the question about the outcome of the first draw. Hence the required probability depends only on the urn's composition at the outset of the experiment. This answer is, of course, wrong. The problem was one of probabilistic *inference*, for which end all the information at one's disposal should be utilized. Thus, the *knowledge* that the second draw resulted in a white ball is relevant, because it removes one white ball out of the possible outcomes of the first draw and leaves us with three equiprobable outcomes, one white and two black. Therefore, the correct answer is $P(W_I|W_{II})=1/3$, just like $P(W_{II}|W_I)$. Students often tend intuitively to think causally. They typically argue: "At that stage (before the first draw), the second draw had not yet been carried out". Furthermore, they insist: "The first ball doesn't care whether the second ball was white or black"; to which one may reply: "Indeed the first ball doesn't care, but *we* do, and mind you: We are no longer 'at that stage', we have advanced beyond it when we acquired the information concerning the outcome of the second draw". This exchange illuminates the distinction between two attributions of uncertainty, one to the external world and one to our state of knowledge (Kahneman & Tversky, 1982). When real world outcomes like those of tossing a coin, playing a football game, or the behavior of a volcano, are considered, the locus of the uncertainty is usually perceived as external. This kind of uncertainty is analyzed causally, hence the disposition of the physical setting to produce the uncertain event is considered. Urns are often effective at producing that type of interpretation (Einhorn & Hogarth, 1981). The subjects' verbal responses, especially their refusal to consider evidence occurring later than the judged event, reflect their reasoning along the time axis. It is congruent with viewing probability as a measure of some feature of the objective urn situation, while the heart of the problem lies in its being addressed to our internal locus of uncertainty. One is asked here to revise one's current state of knowledge in the light of new evidence and to infer backwards. That kind of task could be labeled "diagnostic", rather than "predictive", since one is asked to perform the same kind of judgment as when inferring from symptoms to the underlying disease (Einhorn & Hogarth, 1981). Apparently, those subjects who erred found it difficult to view probability as a summary measure of their own internal state of uncertainty.

A simple way to convince a class of the correct answer, prior to turning to psychological analysis or to problematic concepts like objective vs. subjective probability, is to carry out the experiment. We repeat the basic trial composed of two blind drawings without replacement from an urn, composed as above. We carefully record the outcomes of the first and the second draw. Then we consider all those trials where the second draw resulted in a white ball. Let $N(W_{II})$ be the number of such trials. We count how many of these trials resulted also in white on the first draw. Denote that number $N(W_I \cap W_{II})$. Now one computes the ratio $\frac{N(W_I \cap W_{II})}{N(W_{II})}$ as an estimate of $P(W_I | W_{II})$. The greater the total number of trials, the closer that ratio would be to one third.

Problems with Sons and Daughters.

Mrs F. is known to be the mother of two. We meet her in town with a boy whom she introduces as her son. What is the probability that Mrs F. has two sons? Is it 1/2 because we saw one boy and hence the event in question is that the other childbirth was also that of a male? or is it 1/3, since we learned that Mrs F. has "at least one boy" and hence three equiprobable family structures (BB, BG, GB) are possible, of which our target event (BB) is but one? (Falk, 1978).

Here we have a typical problem where the way by which we obtained the information plays a crucial role. One has to go back to the statistical "experiment" that generated our data to be able to exactly define the conditioning event. It turns out that we met Mrs F. in town with a son. Does this mean that a randomly selected child of the two-children-family was found to be a boy? or does it tell us that the family has got at least one son? The answer depends upon further assumptions regarding that woman's habits. Suppose she randomly selects one child out of the two to accompany her when going out to town (this is the fairest assumption, lacking any other information). In such a case a BB family is twice as likely to yield our observation than is either BG or GB, and a simple Bayesian calculation shows that the probability that the woman has two sons is 1/2 (Bar-Hillel & Falk 1982). Most real life situations from which we learn that a given woman has a son are similarly structured (we either call the house and a son answers the phone, or we come to visit and catch a glimpse of a boy): In all these cases the greater the ratio of sons to daughters in the family, the more probable is the event "a randomly encountered child in the family is a son".

If, however, one interprets the conditioning event as "the family has at least one son", then "two sons" is indeed one outcome of three equiprobable ones, and the required conditional probability is 1/3. Various authors have attempted to invent reasonable "stories" that will fit the latter interpretation of the problem. Gardner (1959, p. 51) simply makes his Mr Smith tell us: "I have two children and at least one of them is a boy". This can be understood as an explicit instruction to treat the subset {BB,GB,BG} as our conditional uniform sample-space (as done in many textbooks, where conditioning events are "given"). Loyer (1983) suggests that we observe the parent at a boy scout meeting, furthermore, in his elaborate story there is a law requiring the parents to attend the boy scout meeting if and only if they have at least one male child. Bar-Hillel & Falk (1982) describe a meeting with a father and son in a male chauvinistic society, where a father automatically selects a son (rather than a daughter) to accompany him, if he has only got one.

The two different interpretations of the problem lend themselves easily to experimental simulation. Carrying out the experiments (or even only the "thought experiments") may illuminate the implicit assumptions, better than the elaborate stories.

Let us prepare four cards: one - blue on both sides - will be denoted BB, where B represents both "blue" and "boy". Another card, GG, will be green on both sides (G - for "green" as well as for "girl"). Two additional cards will be mixed, i.e., blue on one side and green on the reverse one (BG).

In Experiment 1 we shuffle the four cards and blindly draw one of them. Then we randomly put it on the table. Suppose we observe a blue side facing up, what is the probability that the other side of that card is also blue? The experiment can be repeated as many times as one wishes. The proportion of times one turns the card over and finds a blue face on the reverse side, out of the cases with blue face up, will undoubtedly approach 1/2. That is because four blue sides are equally likely candidates to appear as upper faces on the table: The two sides of the BB card and one of each of the two BG cards. In the two former cases the reverse side will be blue, and in the two latter ones - green. This is a perfect model for the story of meeting the lady (who randomly picks one child for going out) with a son.

In Experiment 2 we start by removing the GG card. We shuffle the three cards BB,BG,BG, randomly choose one of them, and observe how many blue faces it has. Now clearly the probability of BB is 1/3. That was a model for the problem stories about the male chauvinistic father, or the parent at the boy scout meeting.

The major distinction between the two interpretations of the problem stems from different sampling procedures. One should carefully note whether we sampled one family of those having at least one son, or one son of a random two-children-family.

Glickman (1982) devised an illustrative urn-model for simulating such problems: Take a family with an older girl and a younger boy as an example. It will be represented by an urn with two tokens, as follows: $\left\lfloor\begin{matrix} G_b & \\ & {}_gB \end{matrix}\right\rfloor$. The urn represents the family. Each token represents one child. The token with G_b written on it stands for a girl (G) and indicates that she has a younger brother (subscript b to the right of G). Similarly, ${}_gB$ represents her brother, a boy with an older sister. The sample space can be written as follows: $\left\{ \left\lfloor\begin{matrix} B_b & \\ & {}_bB \end{matrix}\right\rfloor, \left\lfloor\begin{matrix} B_g & \\ & {}_bG \end{matrix}\right\rfloor, \left\lfloor\begin{matrix} G_b & \\ & {}_gB \end{matrix}\right\rfloor, \left\lfloor\begin{matrix} G_g & \\ & {}_gG \end{matrix}\right\rfloor \right\}$. Now we first sample one random urn of the four (family F. is a random two-children-family), next we sample one token of that urn and notice it is a B token (we met Mrs F. with a son). Note that the experimental procedure guarantees that our B token is equally likely to be one of the four: B_b, ${}_bB$, B_g or ${}_gB$. In two, of the four cases, that boy has a brother. According to the second interpretation (the boy scout meeting), the family is known to have a boy and the sample space is reduced to the three left hand urns. The probability that the family has two boys is immediately seen to be [illegible]/3.

An allied probability problem proved quite perplexing. Suppose you collect in your class, for males and females separately, the number of brothers and the number of sisters each student has. Who will have more sisters, the men or the women? (Mosteller, 1980).

It may seem, at first, that, because "on the average" families have an equal number of sons and daughters, choosing male students for questioning will result in an excess of sisters over brothers, since each respondent does not consider himself. The converse would seem to be true for female subjects. This indeed is usually the prevalent answer intuitively given by subjects (Falk, 1982). However, one of the basic properties believed to govern the mechanism of sex determination is that of independence of the sexes in different births, whether those of siblings or others. Hence, picking one child from a family of n children and recording the number of brothers and sisters he or she has, is equivalent to picking a family of n-1 children and recording the number of sons and daughters in that family. Thus, the expected outcome of the above class experiment would be men and women having equal numbers of brothers and sisters. It is an easy experiment to perform.

If we limit the discussion to families with two children, it can be analyzed in terms of Glickman's (1982) urn device. The unit being sampled now is no longer a family (urn) but a child (token) who belongs to that population. There are eight possible outcomes corresponding to the 8 different types of tokens comprising the urns' contents. If the random respondent turned out to be male he is one of the four types: B_b, $_bB$, B_g or $_gB$, we see that these four men have exactly two brothers and two sisters. The same considerations (and technique) may apply to larger families. Still, some students remain skeptical. Typical doubts were expressed in the words of one of my students (Falk, 1982): "Suppose we were a thousand women in class. Consider all of us together with our brothers and sisters. That large group should consist of approximately equal numbers of males and females. Now, exclude us, 1000 girls, and we'll be left with some more brothers than sisters". She was of course wrong in claiming that the female students plus their brothers and sisters will be equally divided between the two sexes. That group should include more females than males, since daughterless families are not represented among those 1000 families, and furthermore, families abundant with daughters are overrepresented. This kind of bias was introduced by sampling families via their daughters. To demonstrate this we turn again to the urn device (limited to two-children families), suppose all the daughters (G's) in the four equiprobable families (urns) come to a party, and bring with them their brothers and sisters. Each letter in the following list now represents one individual, a capital letter, G, denotes a girl from the class, lower case subscripts denote siblings (older or younger) of these girls: $_bG$, G_b, $_gG$, G_g. Altogether, eight individuals are listed in that group, six girls and two boys. These numbers seriously deviate from equality, and the sampling bias is obvious.

Conclusion

Important psychological insights into the core of probability problems may be gained by experimental simulation. Once one carries out the experiment that generated the problem, there is no room for any hidden assumptions. When one actually samples one item out of a set, one cannot bypass the question whether to randomly choose one family of a specific class of families, or to sample one random child out of a randomly selected family. The different interpretations of the problem are materialized in different sampling procedures. The concept of the "statistical experiment", the outcomes of which define our sample space, becomes meaningful. It is no more just a theoretical construct, to which one eventually pays lip service, but it assumes a crucial role, since it determines the structure of the possible events and their probabilities.

When one gets absorbed in elaborate probability problems, one tends to forget all about the foundations. This is what occasionally leads to misconceptions and paradoxical results. Experimental modeling forces us to newly examine our building blocks, and sheds light on our fundamental concepts.

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Problem Solving: A Correspondence Course

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"I do believe that problems are the heart of mathematics, and I hope that as teachers ... we will train our students to be better problem-posers and problem-solvers than we are." *(P.R. Halmos)*

As part of the extra curricular activities of the Youth Activities section at the Department of Science Teaching in the Weizmann Institute, a correspondence course in mathematical problem-solving was set up. The course is meant for pupils in grades 4 to 9.

The main objectives of this course are to develop problem solving skills, and mathematical thinking through problem solving, to widen and deepen mathematical knowledge, and to search for talented children and encourage them.

Announcement of the course was made in the newspapers. In this way children heard about it and wrote to us about their willingness to participate. Therefore we can say that the main characteristic of this student population was their high motivation and keen interest in mathematics.

Problem sheets were sent to the participants, according to their cognitive level. They sent their solutions to us and received their papers back which had been checked, as well as complete solutions and notes to all problems of their level, and a new problem sheet. After three stages a competition between the best participants was held.

In this paper we explain how the problem sheets were constructed, we demonstrate the difficulties encountered by the pupils, and we take a look at the influence the course has had on the participants.

The Problem sheets

According to Lester: *A problem is a situation in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines completely the method of solution* (Lester, 1978) and *It should be added that this definition assumes a desire on the part of the individual or group to perform the task. Otherwise the situation cannot be considered a problem* (Lester, 1980).

Problems were chosen so as to be challenging at the proper cognitive level, and which require original thinking and the use of non-routine methods for their solution. The mathematical contents include: number bases, percentages, sets, open phrases and open sentences, translation, functions, geometry, tessellations, and topics from number theory.

A lot of problems were given for which the pupil had to "guess" the answer, and then prove that that was really the solution. For example:

1. *Do there exist three consecutive numbers whose product equals 111111111111122? If there are, then write them; if not, explain why?*
2. *Is it possible to express the number 1982 as a difference between two perfect squares? Explain.*

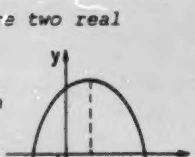
The Construction of a Complex of Problems

One of the guidelines in choosing the problems was to take a set of problems connected either by subject matter or by method of solution. Problems of such a complex were given in several stages. Thus the participants could learn from the comments given on former solutions and from those solutions sent to them. The subjects of the complexes were chosen after examining the difficulties and the needs of the pupils. Below are some examples of complexes.

Problems whose solution by means of functional language is easier than by means of open sentences.

A preliminary examination of 9th & 10th grade pupils showed that many do not see the connection between open sentences and functions, although very often the use of the properties of functions and their graphs make the solution of problems easier. In order to develop such an approach we constructed the following complex of problems for grade 9:

1. *Given the function $y = (x - 1)(x + 1) - (x + 4)x - x^2$, find the vertex of the parabola and construct its graph.*
2. *The equation $ax^2 + bx + c = 0$ is given. It is known that the coefficients satisfy the condition $(a + b + c)c < 0$. Prove that there are two real solutions to this equation.*
3. *The graph which appears to the right represents the function $y = f(x) = ax^2 + bx + c$. Determine whether a, b, and c are positive, negative or zero. Explain your answer.*



4. *It is known that there are no real solutions for the equation* $ax^2 + bx + c = 0$ *and that* $a + b + c < 0$. *Is c positive or negative? Explain.*

5. *Prove that if a, b, and c are rational numbers which satisfy the equation* $|a + c| = |b|$ *, then there are two rational roots for the equation* $ax^2 + bx + c = 0$.

It is hard to solve, for example, the second problem, by means of open sentences. But if we use functional language, we can consider the function $f(x) = ax^2 + bx + c$. From the data we conclude that $(a + b + c)c = f(1) \cdot f(0) < 0$. From here it is easy to complete the argument.

Problems connected with tessellations

Most pupils are not acquainted with the subject of tessellation and are not used to relating to geometric properties by means of tessellations. Therefore we created the following system of problems:

1. *We want to cover a surface with congruent regular polygons in such a manner that vertices only touch other vertices and sides only touch other sides. What kind of congruent regular polygons can be used?*

2. *The Rehovot municipality wants to tile a large public area. It has a great many tiles of two kinds: squares and equilateral triangles. All the tiles of the same kind are exactly the same. Also, the side of the square is the same length as the side of the triangle. For technical reasons it was decided that the vertex of one tile would touch another tile only at its vertex; that is, the tiles would touch each other thus: vertex to vertex and side to side. In order for the pavement to be aesthetic, it was decided that the order of the tiles around each meeting point would be the same. That is, the number of tiles at each meeting point would be the same, as well as the order in which they are laid. The tile-layers could use one kind of tiles or both kinds together. In how many different ways could the area be paved, according to the above conditions?*

3. *As question number 2, but, in addition to the squares and the triangles, there are also tiles which are regular hexagons.*

4. *In a plane there are three regular polygons of x, y, and z sides, having a common vertex. Around this vertex they fill the plane without leaving any empty space. Prove that for each x, y, z that fulfils the above conditions,* $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ *is constant. Find the sum.*

Other complexes dealt with word problems leading to systems of equations in which the number of unknowns is greater than the number of equations and set problems whose solution is easiest by means of Venn diagrams.

Pupils' Difficulties

a) Problems that have no solution

In ordinary classroom instruction, pupils usually do not come up against problems that cannot be solved. Besides the educational value of learning that not every problem has a solution, we saw importance in the fact that checking the result obtained is not only in order to find possible mistakes in calculations, but is an integral part of the solution. Here is an example of such a problem that was given to pupils in grade 7.

A group of children went out on a 4-km hike. They walked 3 km/h for the first 2 km. How fast do they have to walk the other 2 km, so that they would average 6 km/h for the 4 km? Explain.

Many pupils gave 9 km/h as an answer ignoring some of the data. A number of pupils suggested that the children walk the last 2 km. in zero time or with infinite speed. About a third of the 34 pupils stated correctly that there was no solution.

b) Problems including transfer from an everyday problem to a mathematical one.

The problems in the course were such that their solution strategies were not evident from their formulation, so high-level thinking was required to solve them. Here are two examples of problems formulated as everyday situations, in which the main difficulty is their reformulation as mathematical problems, that is, the construction of a suitable mathematical model.

Problem A:

In honor of Israel's Independence Day, each city sent greetings to its nearest neighboring city. Supposing that the distances between cities are different, prove that each city got no more than 5 such greetings for Independence Day.

Problem B:

A baker who owns 2 bakeries decided to build a flour storeroom to be used for both. Where should the storeroom be built so that transporting the sacks of flour from the storeroom daily will be cheapest? (Suppose that the daily requirement of flour for each bakery is constant.)

The first difficulty in such problems is the construction of a suitable mathematical model for the problem which is expressed as an everyday situation. Problem A can be expressed as a geometric problem, and Problem B can be expressed as a problem of finding a minimum of a function in one variable.

One of the difficulties which arises for pupils who try to solve the problem without first giving it mathematical expression, is that they base themselves on arguments which are either not exact mathematically or unsuitable for the given situation. For example, in Problem B, it was argued that if the relation between the consumption of flour by the bakeries is 1:2 then the storeroom should be built 1/3 of the distance between them, closer to the one that uses the most flour. This phenomenon of giving an irrelevant argument appeared in Problem A as well.

In Problem B there is an additional difficulty - the "missing data". Pupils that tried to solve the problem without transferring to an exact mathematical formulation ignored the problem of the location of the storeroom and made additional suggestions for reductions in cost, such as paying less to the drivers, building a cheaper storeroom etc.

Effect

One of the aims of the course was to encourage pupils to occupy themselves, on their own, with mathematics. From the letters received from participants, it seems that many tended to involve others in their immediate environment (parents, teachers, neighbours etc.) in solving the problems. Some of them ask for bibliographical references so they could find more extensive material on the subjects presented in the problems.

Another way of judging the effect is by the feedback from checking their answers and from sending them our answers. For instance, let us analyze the effect of the complex of problems and solutions on the subject of set problems whose solution is easiest by means of Venn diagrams.

In the first stage in grade 4 the first problem we gave had as its main difficulty the finding of the number of elements of a sub-test when two sets are given whose intersection is not empty. In the next stage we sent the participants the solution of the problem and the second problem, in which the degree of difficulty was much greater than in the first. In this second problem, not only were there three sets instead of two, but the second subtraction stage was to be done on the first subtraction. In order to check whether there is an effect of the feedback that the pupils got from us, we gave, in the first stage of grade 5 a problem equivalent to the one in the second stage of grade 4. This question was given without any preparation. It seems to us that the simplest method of solution is by Venn diagrams, and in fact 42% of the 36 grade 5 pupils who solved this problem (got at least 8 out of 10 points) used Venn diagrams for the solutions and 94% of those who chose to use Venn diagrams solved the problems. For this reason, we added a solution using Venn diagram to the solution for stage one of grade 4. In their answers to the second problem, involving 3 sets, we found that it had influenced the pupils. Whereas among the 118 grade 5 pupils (who had not received.guidance) only 14% used Venn diagrams, 46% of the 79 grade 4 pupils used this method. (In the first stage not one of them had used Venn diagrams). Furthermore, 81% of the grade 4 pupils who succeeded in solving the above problem used Venn diagrams.

Conclusion

From our experience, the course enables pupils from all over the country to occupy themselves in a field that interests them. The personal contact, by means of checking each and every solution, with comments made for each answer helps to create effective feedback for the participants. The gradual construction of problems such as those shown here seems effective and helps to achieve the aims of the course. The difficulties encountered by the pupils must be followed in order to construct the systems of problems for the years to come in the most beneficial manner.

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F. COMPUTERS AND MATHEMATICS LEARNING

COGNITIVE STYLES IN THE FIELD OF COMPUTER PROGRAMMING

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1. Introduction

Since many years we are dealing with the problem of the utilization of different representative levels (enactive, iconic, symbolic) for the formation of algorithmic concepts for pupils, needed in the field of computer programming - first we did this through the development of didactic materials and lessons (Cohors-Fresenborg et. al 1979a, 1979b, 1982a). The success of these didactical efforts has led us to the intention to develop the original ideas of them into an empirical didactical research oriented on a cognitive psychology. First of all we had to define the role of representative levels in forming algorithmic concepts. Further if there are different cognitive styles in the construction of algorithms, and if the pupils generally have a preference for different techniques in constructing algorithms. Moreover our experiences in teaching showed that pupils are differently talented for the performance in constructing and analysing algorithms. Herein we parted from the premonition, which we thought to be teoretically plausible, that such performances depend on the knowledge of logical and elementary actions. Therefore the enactive representations are especially important in the field of algorithmic elaborations.

The didactical materials which we elaborated enable even young pupils without any knowledge to deal with difficult mathematical problems in algorithms within a short period of time. The computing machine "*Registermaschine*" for demonstration purposes, was especially designed for the mathematical aspect of algorithms, after a short introduction it is very easy to handle by the pupils. For that reason we didn't choose one of the usual calculators basing on known programmation languages.

As we intended with our researches to recognize, to explain, and therewith to understand the thinking processes of the pupils, a design of pretest-posttest oriented on behavior seemed to us not to be helpful. On the other side, we thought that the simple observation of an open problem-solving situation and its following interpretation could complicate the evaluation and last not least the comparison of such a free relation between pupils and teacher. Furthermore it must be questioned, in which way the results of the analysis of a single case might be generalized to obtain a theory for other cases. But in this paper we will not discuss the general problems of methodology in empirical research in the field of mathematical didactics (see Cohors-Fresenborg 1983).

In spring of 1981 we startet an investigation with the following questions: Is it possible to classify pupils according to their performance in constructing algorithmic concepts? 14 pupils of grade 7 in a secondary school participated in the experiment, where they had to elaborate algorithms on different representative levels. They did not have any knowledge in this field before. Separately they worked in 6 units of 45 min. each, on two different kinds of problems, which we named as *constructive* and *analytic* problems (The reader may find a report of the investigation and different examples for the problems in Cohors-Fresenborg 1982b).

The questioning of the problem, its strategy of solving and the result are quite different in case of constructive and analytic problems, for the solving of which we suppose different cognitive styles of thinking. The constructive problems can be commenced on different representative levels: by developing a sequence of basic activities with sticks, by developing a calculation network and by determinating a suitable programword for "Registermaschine". We observed those different levels of treating the problems, which could be a hint for the different types of representative perceptive faculty. Two types can be differentiated in the so-called analytic problems: the determination of a suitable term of function, calculated by the given word of "Registermaschine", presupposes a semantic comprehension of the programming word can be proved by the pupil giving a suitable function of the number of steps.

2. Critical reflection of the first pilot study

We had the target to document the process of thinking of pupils by the relation between teacher and pupil. Primary this should be shown by the given hints and the reaction of the pupils thereon.

The analysis of the videos after the examination showed clearly the following debilities:

1. The presupposed aids where insufficient. The analysis of the videos shows more dialogues between teacher and pupil than originally planned. That means that only a very small share of the real difficulties was supposed to emanate while planning.
2. The given aids directed the pupils close to a very programmed method of solving the problems. As the videos showed, some pupils had other ideas for the solution than the presupposed. The aids given by the teacher sometimes distanced the pupils from their own ideas and led them to the planned process.
3. The teachers reacted differently in the case of each pupil. Apart from the aids counting for the analysis, the teachers gave instructions, which

were at most different from case to case.

4. All aids were valued equally and listed in the analysis. The needed aids had different characters, e.g.:
- emotional hints, which should encourage the pupils to begin a task,
- hints, correcting mistakes,
- hints, showing analogies to already solved problems,
- units reminding already known but forgotten facts.

The above mentioned critics let us change the design of the main investigation in order to increase the quality of the same. Although, we wanted to keep the idea of constructing a network of aids for each problem with the aim of documenting the thinking process of the pupil by given i. e. not given aids.

The structure of aids we should select had to enable the teacher much more than before to a classification into categories during the test: i.e. the decision whether an aim has been reached; moreover, to assume the momentaneous thinking of the pupil and to decide whether an aid is necessary. These decisions have to be possible for the teacher, even if during the analysis the comparison of the different aids, given by the teachers is more difficult.

3. Second pilot study

Our reflections show that the aids to be given depend on the following conditions:

1. condition: The aids shall reflect the *learning process of the pupils* and not that of the teachers. Therefore the different strategies of solution of the pupils should be taken into account. The aids shall not oblige to use a programmed method.
2. condition: The aids should make possible a diagnosis about the learning process.
3. condition: The different aids should be given by all teachers in all tests for the same reason.

The second question we discussed was: Is it possible to use a computer, programmed to react on a code-word given by the pupils by searching the corresponding aid and transmits it to the pupil? We thought this proposition useless because of the following:

A computer is only able to choose an aid according to sintactical points of view. An instructed teacher, who is able to guess hypothetically what is in the pupils mind decides, however, following semantical points of view. The following decision results from the mentioned conditions:

An aid has the following form:

(*number*, *diagnosis*, *comment*, *aim*, *formulation*)

The *numbers* of the single steps of aid makes it easier for the teacher to take into account the different strategies of solution of the pupils, because he is not forced to follow their order.
The *diagnosis* enables the teacher during the test to decide whether, when and how to help the pupil.
The *comment* is closely connected with the diagnosis. The aid to be choosen shall result clearly from the pair aim/diagnosis. Herein the diagnosis and the aim to be reached have priority over the *formulation*.

We thought that such a system could help to harmonize the aids in its position and efficiency with the supposed thinking process of the pupils. By this way it is largely possible to prevent the teacher forcing a certain way of solution.

For the analysis there were fixed in advance some points of diagnosis in the supposed way of solution, which should serve as qualitative evaluation of the solution style.

In the relation between teacher and pupil exists a strong component of non-verbal communication due to the quantity of activities to be observed. The non-verbal communication can be determined as a development of the proposals given by SCHWANK (1979) for the improvement of the method of thinking aloud.

Even if we achieve to document as close as possible the process of solution of a pupil in this way, there still remains the problem of comparing the performance according to the enumeration of the needed aids. Nevertheless not all the aids are to be valued equaly, but due to its large number (up to 40 in one problem) there results a compensation, enabling a rough classification of the quality of performance.

4. Resume and prospect

We decided to solve our problem, to find a satisfying description of the handling of concepts in the field of computer programming, by using a "structure of aids". The first simple applications showed to be usefull in order to achieve a certain standardization of the pupils way of solving problems.
One of our next aims is to satisfy a higher demand of precision. Furthermore it must be prooven in how far theories of structurized networks, developed

in the field of an automatic grammatical analysis, can be used for a suitable description of the different steps of the processes of understanding and solving problems. In the case of an existing "diagnose" and "aim" it is possible that the teacher, gives an "aid" to the pupil, by indicating him a change in his network of thoughts and e.g. showing a new line of relationship. From the experience with certain underdefined expressions results a "prejudice", which is ever tested in the reality, the corresponding network of expressions is reduced, changing its structure.

We want to point out, that it is not our main task to formalize structures of knowledge and its handling, but particularly to include into the network our points of view about the specific nature of processes and performances of thinking (see VERMANDEL et.al. 1979). A typical way of mathematical thinking is the transformation of products of thinking processes into objects of reflection. Thereon, the following possibility is requested: to follow up this idea, so that after finishing a process of understanding and abstracting, an object is only present as a knot in a further network etc.; perhaps with a dynamic network. So we would dispose of a possibility of expression, which would not only be useful in the field of computer programming, but also be applicable to other mathematical branches and guide to further consequences.

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USING A MICROCOMPUTER TO TEACH REPRESENTATIONAL SKILLS

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ABSTRACT

A program has been developed that allows children to use a microcomputer rather than physical objects to solve word problems. They produce pictorial sets of objects one at a time and can make a single set, or make two sets, or remove elements from a set they have constructed. The connection between informal modeling processes and the formal mathematical symbolic representations is made by teaching the children that they do not have to contruct sets one element at a time ; they can construct them by writing number sentences. Results of a pilot study carried out with four first-grade children indicate that the program is effective in teaching representational and problem solving skills.

BACKGROUND

In the last few years a substantial body of research has focused on the learning of addition and subtraction concepts in general and on the solution of addition and subtraction word problems in particular (Carpenter, Blume, Hiebert, Anick & Pimm, 1982 ; Carpenter, Moser & Romberg, 1982 ; Riley, Greeno & Heller, 1983). Currently there is good agreement regarding the basic characterization of addition and subtraction word problems, and there is a reasonably consistent picture of the difficulty level of different types of problems and the informal problem-solving strategies children invent independently of instruction. However, relatively little is known regarding the transition from these informal strategies to the formal addition and subtraction skills taught in school.

A key aspect of the transition from solving problems using informal procedures based upon simple representational skills to a formal mathematics approach is writing mathematical symbols to represent the problem and its components. At the time children are first introduced to writing mathematical sentences to help solve word problems, their informal strategies and procedures make more sense to them. As a consequence, they see no connection between the two activities. The operations represented by the number sentences are often inconsistent with the modeling and counting strategies used to solve the problem. Writing a number sentence is something that young children do for the teacher, something they often perceive as unerelated to the solution of the problem.

In a study investigating the effects of initial instruction on the processes children used to solve basic addition and subtraction verbal problems, Carpenter, Moser and Hiebert (1981) considered the role of writing number sentences in the solution process. Prior to instruction 43 first-grade children were individually tested on a variety of addition and subtraction word problems. After a two-month introductory unit on addition and subtraction, the children were retested. On the posttest most children could write number sentences to represent addition and subtraction problems. However, very few recognized that the arithmetic sentence was a mechanism that they might use to help them solve the problem. Once they had written a sentence, most children appeared to ignore it and used the semantic structure to decide on a solution strategy. The fact that sentence writing did not influence children's solution processes suggests a lack of coordination by them between the two processes.

DESCRIPTION OF THE COMPUTER PROGRAM

The major feature of the program is the ability to enter onto a video display pictorial and symbolic configurations. The display is arranged in three adjacent sectors which can be thought of as corresponding to the elements of the number sentence a + b = c or a - b = c. Pictorial configurations consist of small squares arranged in a pattern resembling the TILE configurations used in Japanese elementary education (Hatano, 1982). See Figure 1 for an example of the configuration. Entry of the pictorial configuration occurs in two ways -- a one-by-one incremental entry effected by depression of the $\boxed{\rightarrow}$ key or by a depression of appropriate numeral key(s). Space limitations confined the number of squares in a sector to a maximum of 30. Squares appear only in sectors a and b. Removal of squares from a sector is carried out by depression of the $\boxed{\leftarrow}$ key. The "take-away" action can be simulated by entry of squares into sector a and then movement of a subset of those squares to sector b, such movement being effected by initial depression of the - (minus) key followed by depression of the $\boxed{\rightarrow}$ key as many times as required or by depression of the numeral key.

Standard mathematical symbols for numbers, operations, equality, and an unknown quantity (□) can be produced by depression of the appropriate keys as needed. These symbols

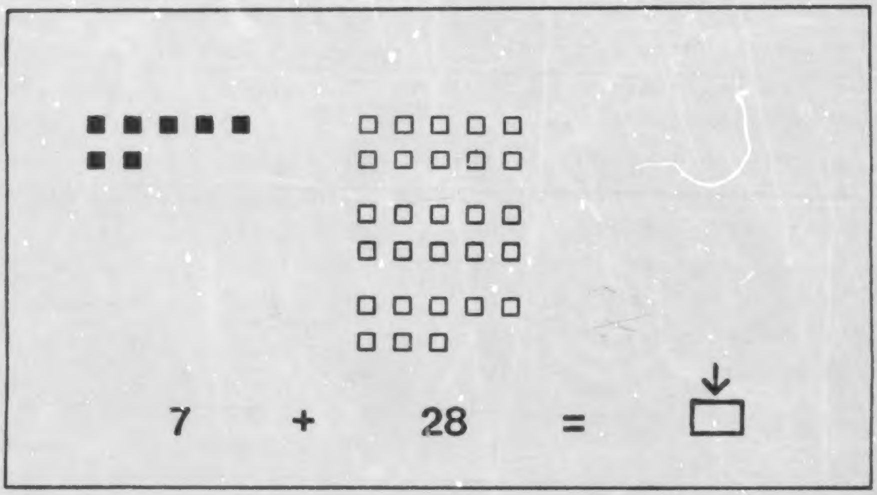

Figure 1. Video screen display after entry of 7 + 28 = □.

appear in the lower one-third of the video display as indicated in Figure 1. The program calls for the production of only numerals, or for a complete number sentence that is essentially correct in form. Incomplete sentences such as $5 + 7 \square$ or $5 \quad 7 = \square$ would not appear. Sentences that are impossible to solve within the domain of whole numbers such as $13 - 15 = \square$ or $9 + \square = 3$ would not be accepted by the computer. Sentences, either open or closed, of the form $\underline{a}\ \underline{+}\ \underline{b} = \underline{c}$ may be entered. A complete description of, as well as listing of the program may be found in Moser and Carpenter (1982).

THE TEACHING EXPERIMENT

Four first-grade children, two male and two female, of middle range of ability participated in a five-week teaching experiment in April and May, 1982. Individual interviews were conducted with each subject in which the child was asked to solve the following 10 verbal problems : Join, Separate (2), Compare (add), Compare -- find the difference (2), Join -- missing addend (3), and Separate -- missing minuend. For one each of the Separate, Join -- missing addend, and Compare -- find the difference, the child was also asked to write a number sentence prior to solving. Several tasks were also administered to assess each child's ability to use counting-forward procedures.

Each child received nine individual lessons conducted by a researcher. A second adult observer/recorder was present during all lessons. Initial lessons involved an introduction to the computer and production and manipulation of pictorial sets without any accompaniment of appropriate symbolism. Next came the production of sets by means of depression of numerical keys. The third step in the learning sequence dealt with the production of complete closed sentences as representations of solution procedures. The final step involved the "writing" of open sentences with a as a representation for the missing quantity. Part of each lesson included a presentation of a variety of verbal addition and subtraction problems to be solved by the student with the aid of the microcomputer. Essentially the student was asked to use the features of the computer program that were currently being taught. On the day following the last lesson, a follow-up individual interview was conducted with each student as a posttest. Six problem tasks were presented with the computer available to assist in sentence writing and solution. An additional six problems with the same semantic structure as the first six were given with paper/pencil and physical manipulatives to help with sentence writing and solution.

RESULTS AND DISCUSSION

Prior to instruction, all of the four experimental subjects wrote inappropriate number sentences for all but the most straightforward addition and subtraction problems. Furthermore, they generally viewed the number sentences as unrelated to their solution processes and ignored the sentences they wrote when solving the problem, arriving at a solution by directly modeling the action or relationships described in the problem.

Following instruction all four subjects could write number sentences to represent most problem situations and successfully used this ability to solve a variety of problems using the computer. Three of the four subjects transferred this ability to problems without the computer. To solve a simple word problem, they would first write a number sentence and then use a solution process that modeled the number sentence not the structure of the problem.

One of the factors that significantly facilitated children's ability to represent and solve certain word problems was instruction on writing noncanonical open sentences (e.g., $5 + \square = 13$ and $\square + 5 = 13$). These sentences allow children to write number sentences that are consistent with the semantic structure of missing addend problems. It has been clearly documented that young children solve missing addend problems using an adding on or counting up process which is most closely represented by an open sentence of the form $a + \square = b$ (Carpenter & Moser, 1982). During and following instruction, all four subjects consistently wrote noncanonical sentences to represent missing addend word problems. In fact, they also wrote sentences like $\square - 4 = 7$ to represent missing minuend problems, even though they received very little instruction in this type of number sentence. Research suggests that initial instruction on addition and subtraction should include noncanonical sentences. The results of the pilot study strongly support this conclusion.

The improvement in representational skills, however, was not totally a function of learning about noncanonical number sentences. Following instruction, the children were also generally more consistent in writing appropriate canonical sentences for a variety of problems and using these sentences as a basis for solving problems. Students' performance during the lessons also supports the conclusion that the instruction was successful in developing representational skills and helping the children understand the relationship between their informal strategies and the formal mathematical representations. Children quickly grasped the concepts presented to them and were almost immediately able to use them to solve problems. With regard to the mechanical aspects of the children's interaction with the computer, the results can be characterized unambiguously as positive. All four first graders demonstrated their ability to work with an unfamiliar machine without any difficulty. No mechanical or motor coordination problems were detected.

In conclusion, the computer appears to allow children to rely upon their informal mathematics in an area of formal mathematics such as sentence writing. The use of verbal problems does seem natural to young children because they are able to solve them in their own informal ways. The present experiment demonstrates that the computer may allow them to represent those problems in a formal way, even though they have not yet completely learned the formal algorithms and number facts. These findings suggest that instruction could be changed to make better use of children's natural ability to solve verbal problems in learning the formal mathematics of addition and subtraction. This pilot investigation suggests that the microcomputer can have an important role in that instruction.

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SOME PROBLEMS IN CHILDREN'S LOGO LEARNING

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1. INTRODUCTION

This talk is an informal, personal summary of three years' experience of working, and observing other people's work, with children and computers in the Logo method. (By the "Logo method" I mean the whole learning environment, including the computer language as well as the educational philosophy and practice, as described in Papert, 1980.) While I basically remain enthusiastic about Logo's vision of children's interaction with the computer, and the potential of this approach to foster deep and meaningful learning, I am now also aware of some fundamental difficulties on the child's route to acquiring the "powerful ideas" as envisioned by Papert. The current literature contains excellent books about Logo's educational *vision* (Papert, 1980) and aspects of the computer *language* (Abelson, 1981), and most subsequent writings elaborate on the same themes (e.g., Byte, 1982). In contrast, little has been explicitly written on *difficulties* in the method, even less on attempts to overcome them. (The Brookline Project Report (Papert et al., 1979) occasionally touches on some of these themes, especially in the detailed reports of the children's work.) If we are to realize the great potential that the Logo method holds, it is urgent, in my opinion, that the Logo community turns its full attention to studying these difficulties and the educational problems they pose.

Invoking Logo's central theme that a complex project cannot be expected to run perfectly the first time around, nor should it be abandoned because it has been found to contain "bugs", my suggestion is that the time has come to start a careful "debugging" of some aspects of the Logo method. We do not know much at present about how to actually fix the bugs – this task needs much more research. My talk will remain mainly within the first stage of the debugging activity, that of *identifying* the bugs. The remarks in the rest of this abstract relate mainly to children in the sixth grade (about twelve years old), but since the observations include some rather bright students, one can expect some of these problems to occur with older children as well.

NOTE. Since the literature about Logo mainly elaborates on the merits of the method, this exposition stresses the difficulties encountered, thus running the risk of painting an unbalanced picture (especially for readers with no prior acquaintance with Logo). The reader should therefore bear in mind that the following observations are meant as suggestions for further research in Logo rather than criticism. In fact, it is my belief that, with more research, one can devise ingenious educational means to solve many of these problems.

ACKNOWLEDGEMENT. It is a pleasure for me to acknowledge the profitable and pleasant collaboration with Pearla Nesher and Rina Zazkis.

2. "PIAGETIAN LEARNING" AND POWERFUL IDEAS

Perhaps the most basic difficulty I have encountered is the apparent conflict between the ideal of spontaneous, non-directed, "piagetian" interaction with the computer, and the acquisition of some of the "powerful ideas" envisioned by Papert. (The terms "Piagetian learning" and "powerful ideas" have special meaning in Logo. See Papert, 1980.) Many Logo teachers interpret Papert's writings as discouraging any direct intervention (such as prescribing certain tasks chosen by the teacher, or other curriculum-type activities), except for responding to the child's initiatives. However, my observations indicate that under such conditions, most children tend to fall into a "hacking" kind of programming which does not seem to be conducive to learning deep and sophisticated ideas. This style is characterized by a great amount of trial-and-error activity accompanied by little planning or reflection; and it often enables children working on turtle-graphics projects to achieve their goals, or change them in the middle, without fully understanding the nature of the bugs they have encountered and how they were eventually eliminated. Programming in this style may be justified, perhaps even recommended, in the case of students with special learning difficulties; but we see it happening even in cases where relatively bright twelve-year old children are taught Logo for several months by relatively bright, sensitive and responsive teachers. Learning powerful ideas requires the development of sophisticated new intelectual structures in the child's mind, with the necessary concurrent elevation in level of understanding and reflection. Could it be, then, that in order to achieve this goal the child needs a bigger "push" (perhaps in the form of more planning and directing of her activity) than we are usually willing to excercise in Logo? If this turns out to be the case, then we must face the educational challenge of finding more active ways of helping the child, without sacrificing the spirit of meaningful and exploratory learning.

NOTE. Papert's vision of "Piagetian" acquisition of powerful ideas is very appealing. It is based on an analogy with the way young children learn their natural language and early mathematical concepts. If this kind of learning does not quite happen in the Logo environment, as the preceding observations seem to indicate, it should be very interesting to speculate on the differences between the two situations.

3. SUBPROCEDURES AND MODULARITY

Most children seem to pick up quite easily and naturally the first steps of Logo programming — navigating the turtle and writing simple procedures (that is, entering the editor, writing text, exiting the editor and running the procedure). The next major step, however, already poses serious problems. I am referring to the use of subprocedures, modularity, and the whole cluster of intelectual tools that go under the name of "structured programming". When talking about learning the use of subprocedures, one must distinguish two levels of such learning. Superficially, the child is simply learning a set of syntactical rules (for instance, the knowledge that child-defined procedures can legally "call" each other). This seems to pose no special learning difficulties. The deeper aspect has to do with the creation of a conceptual framework that brings the child to actually use these tools willingly and successfully on her own; it requires the ability to conceive of a complex procedure as a hierarchy of subprocedures with "interfaces" between them, as well as the tendency to do so spontaneously. This ability appears to be more sophisticated and difficult to achieve than we have previously suspected. In fact most six-graders I have seen tend to write long, step-by-step, unstructured procedures. Moreover, even when explicitly prompted to use subprocedures they seem to "resist" the suggestion and return to their "linear" style as soon as they are left alone. An ongoing research (with Nesher and Zazkis) indicates that one major source of difficulties is the lack of a clear concept of the *interface* between any two subprocedures, and the related importance of the turtle *state* before and after each subprocedure. (Further explanations and examples will be given in the conference talk.) The difficult educational question is, again, how to help the development of this conceptual framework in the child's mind.

NOTE. In the Brookline Project Report (Papert et al., 1979) there seems to be no clear distinction between these two levels of learning to use subprocedures. The deeper level is probably meant when this skill is considered one of the conditions for saying that the students "have 'learned to program' in any significant sense" (p. 1.14), while the more superficial level is probably meant in the following quote from the Summary of Findings (ibid): "Which students learned to program? (...) all students except those in the lowest quartile of school performance did reach our criteria." (My conclusion is drawn from the detailed descriptions of the students' work. The relevant criterion is (A3) on p. 1.5).

4. MATHEMATICS

The issue of learning mathematics in Logo is very complex, and I shall not go into it at any length. While an expert watching the kids work in Logo can clearly see a lot of mathematics going on, it is not clear how much of it is actually learnt by the children. In other words, the turtle may be a "math-speaking creature", but we cannot automatically assume that the children always listen to what it is saying. As in the case of other powerful ideas, it is possible that a more structured environment is needed if the children are to *acquire* these mathematical contents. In any case, it is important that when reporting on children's learning of mathematics in Logo, a clear distinction be maintained between the mathematics observed by us, and the mathematics actually acquired by the child. For example, while "there is a lot of group theory going on" in children's Logo work (Leron, 1982), one could not sensibly claim that they have actually learnt group theory. The mathematics we see in the Logo environment is mathematics *potentially* learnable by the children; making them actually learn it is again a major educational challange for Logo.

5. OTHER ISSUES

In this section I shall briefly review some more topics whose degree of sophistication seems to require a more structured learning environment than is usually associated with Logo. As in the case of subprocedures (Section 2), the learning of each of these topics has both shallow and deep aspects. The deeper aspect is the one associated with "powerful ideas", and is the one that will concern me here.

5.1. VARIABLES, RECURSION. A non-trivial understanding of these topics requires a degree of formal thinking that is not easy for sixth-graders, and good concrete models, such as Mail Boxes for variables and Little People for recursion, are very helpful. Variables appear most naturally as inputs to procedures, where they control easily observed graphic properties, and should probably be introduced this way. (The assignment command MAKE can and should be postponed till much later.) However, issues concerning outputs, scoping and the passing of variables between procedures are complex and possibly too difficult for most sixth-graders. Recursion should be approached with particular care, as it conceals very sophisticated and complex structures behind simple and innocent appearances. Many of the children I have taught or observed, tend in such cases to learn by rote certain "idioms" and the effects they produce, without really understanding how they work. While this may be a natural state in the first stages of a new topic, it makes good planning and debugging almost impossible, so its persistence may later inhibit progress.

5.2. DEBUGGING. It is well known that long, unstructured procedures are hard to understand and debug. Somewhat unexpectedly, many of our beginning sixth-graders seem to find modularized, "structured" programming just as hard. This is important, as it makes the task of "converting" them to the latter style of programming harder for us. One possible explanation is that the well known advantages of structured programming depend on a fuller understanding of the process than our children usually possess. Simple step-by-step programs, on the other hand, may be easier to debug (at this early stage) by mere "hacking". In addition, the above-mentioned research (with Nesher and Zazkis) have underlined two specific debugging-related difficulties: One (already mentioned), a lack of clear concept of the interface between subprocedures, and the role of the turtle-state in determining this interface. Two, the difficulty children find in formally and rigorously "playing turtle" (that is, tracing the steps of the procedure precisely as written, without interference from the child's intentions and intuitions). In fact, one of the topics under study in that research is the possibility of specifically training the children in these skills through interesting and carefully thought out assignments.

5.3. BEYOND TURTLE GRAPHICS. The non-graphic parts of the Logo computer language are, in my experience, mostly too difficult for sixth-graders. While they can learn some of the commands regarding list-, text-, or music-processing, most interesting projects would pose serious difficulties. However, graphics programming in Logo, and the ideas that can be learnt with it, are rich enough to easily occupy the children for a whole year's course, by which time they may be ready to move on.

6. CONCLUSIONS

I wish to conclude with an optimistic note. The problems described in this article have all been observed in cases where teacher intervention had been minimal, with very little planning or directing of the child's activity. Just as significantly, the time span of the children's Logo work had been rather short: usually no more than four months, or equivalently, about thirty-to-forty hours at the computer. In contrast, one envisions a well-planned, spiral course extending over perhaps several years, based on a fuller and more detailed understanding of children's learning in the Logo environment, and accompanied by well-written learning guidebooks. The development of such a course is a non-trivial task if we are to preserve the spirit of exploratory and personally meaningful learning. But it is quite reasonable to expect that such a course will have successfully dealt with these problems, thus greatly increasing the intellectual benefits children derive from their Logo work.

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LEARNING STATISTICS WITH THE HELP OF THE COMPUTER

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1/ The Conservatoire National des Arts et Métiers is a University devoted to research and teaching in the field of technology and Sciences as applied to work situations. Students are already engaged in professional life and attend evening classes. On the average they are about 30 years old and their previous education ranges from Secondary School level to the Master's degree. About 120 students in Human Sciences attend each year an optional Course in Fundamental Statistics, followed by an optional training session of exercices and problem solving done with the help of microcomputers. The computers are used for practicing statistical calculations and statistical reasoning in many different situations.

A research team called "Didactique de la Statistique" has been created in the Laboratory of Work Psychology with the aims of :

First, analysing the specific difficulties of adults in learning statistics and in using statistics for a more general reasoning process that includes both the analysis of experimental data and decision making ;

Second, initiating more fundamental research on the epistemology of statistics ;

Third, developing teaching and learning aids of different kinds : programmed learning booklets, case studies, games, and computer programs with their pedagogical handbooks.

This presentation will deal only with the computer programs.

2/ Description of the learning situation

For this experiment we used the computer facilities which are available in french secondary schools : 1 mini computer T 1600 Télémécanique connected with eight terminals (alpha numeric displays) ; and since 1982 : eight microcomputers with alphanumeric and graphic display (micral 8022 G, R2E Company).
The programming langage is L.S.E. (Langage symbolique d'Enseignement).

Twelve programs have been proposed each year to about 100 students, allowing 60 hours of interactive learning, with no need for anykind of knowledge in the area of computer systems or progamming languages. A Three hours period each week are devoted to work with the computer.
Two or Three students work together on each microcomputer and this small-group situation appears to play an important role.
The teacher is present in the computer classroom and is at the students'disposal for any request, exploration, or help.

Of the twelve programs, seven deal in statistical description :

. class distributions and histograms (with equal or unequal intervals : HISTE and HISTI) ;
. relevant statistics for the different types of variables : OPERA ;
. estimation of median, mean and regression value : MEMOR ;
. bivariate frequency tables : TABLO ;
. practicing the estimation of the correlation coefficient : CORRE ;
. calculation of the Bravais Pearson coefficient : CORRU ;

The other five deal in statistical inference :

- . sampling distribution of the mean : DIMOY ;
- . sampling distribution of a frequency : DIFRE ;
- . the confidence internal (of a mean, of a frequency, of the difference between two means or two frequencies) : CONFI ;
- . chi-squarred test for a discrete variable : comparison of an observed distribution with a theoretical distribution : KHI2D ;
- . chi-squarred test for a numeric variable : comparison of an observed distribution with the normal distribution : KHI2N.

3/ Different types of interactive learning dialogs

None of the programs was conceived on the basis of programmed learning or questionnaries, but they play various cognitive roles depending on each specific topic. For exemple, some dialogs help the student to acquire intuition of specific statistical concepts without having to understand the mathematical demonstration (MEMOR), or to make all the necessary calculations, tables or graphs, which is an easy but long and repetitive work (CORRE, TABLO).

Other dialogs make clear some subjective aspects of data analysis, like class distribution (HISTE, HISTI), and still others help the student in making relevant choices when analysing data for one or several variables (OPERA, TABLO) ;

Simulation games give students the opportunity of "playing" with a model (DIMOY, DIFRE and CORRE) : they put data and parameters into the machine, the program performs calculations which couldn't be done by the student. Then the student is presented with the results of his choices, makes the relevant interpretation, and tries again until he has obtained a precise representation of the model.

Additional dialogs provide students with guidance in solving complex problem (KHI2D, KHI2N) : they help the student to learn the operations and sequence of an elaborate algorithm.

Finally, one dialog provides the student with display and calculation, but requires him to estimate the computerized value of the statistics himself before it gives the result.

4/ Each dialog has been designed so as to adapt to the teacher's and to the learner's needs :

Each dialog comes with a manual containing one or several specific problem(s) to be solved.
Also included with each problem are the aim of the experiment or inquiry,
the data collected for that purpose,
a list of questions on the data and for each question precise indications on how to use the program to answer them.

The teacher can use the program with the manual as it is, or use it for many different purposes depending on the problem he gives to the students and the type of questions he asks about the data.

Furthermore, the training session with the computer can be organized and directed in a more or less precise way, depending on the autonomy of the student (and usually in such a way to increase this autonomy).

For exemples the first time he is using the program, the student generally uses the manual or another document made by the teacher ; later on he is asked to solve a given problem with given data but he is supposed to define relevant questions and to find out the appropriate program(s) to use ; finally the students use the programs for analysing their own experimental data.

5/ The results of the 5 year experiment and conclusions

About 80 students per year took part in the computer sessions, although they were optional, and very few of them gave up. Most of them said they enjoyed working on the computer very much.

The work done by each student on the computer was not recorded, but the students were asked to write down a report for each session.
The analysis of the reports led to the conlusion that using the dialogs makes the students perform better in the general procedure of data analysis, as well as in the use of complex algorithms. It also provides them with a deeper comprehension of the meaning of statistics and procedures (compared to the "no-computer situation").

Finally solving problems in small groups gives students the opportunity for discussing the questions, justifying their choices and interpreting results.
Working with the computer generally decreases anxiety and increases performance ; learning originates in both types of dialog : those with the computer and with other students.

VISUALIZING HIGHER LEVEL MATHEMATICAL CONCEPTS USING COMPUTER GRAPHICS

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ABSTRACT

The computer is going to revolutionize mathematical education, not least with its ability to calculate quickly and display moving graphics. These facilities have been utilized in interactive programs to demonstrate the ideas in differentiation and integration, evoking new dynamic concept images.

THEORETICAL BACKGROUND

The work described in this paper is the result of a happy accident of history. Over a number of years mathematics educators have studied the concept imagery generated by students when learning the calculus and now microcomputers have become available which can draw moving pictures to provide powerful cognitive support for this imagery. Though by no means a total solution, it is hoped that interactive work on the computer can give fruitful insight into the calculus that is potentially more meaningful.

The research of Orton [3] confirmed that a group of students taught by current methods in the U.K. had great difficulty with a number of ideas in the calculus requiring relational understanding. These included the idea of rate of change between two points on a graph with all the possible signs involved, the notion of the derivative as a limit, the idea of the area as the limit of a sum and the meaning of positive and negative areas.

Other authors have noted interference in mathematical meaning through the use of words that have different colloquial connotations. For instance, the idea of a "limit" being unreachable (Cornu [1]) or the term "gets close to" carrying the implication "not coincident with" (Schwarzenberger & Tall [4]). Ervynck [2] has also documented problems with limits and suggests the value of pictures to visualize the processes involved. Standard pictures found in text-books have two major problems: they are static, and so fail to fully convey the dynamic nature of many of the concepts, and they also tend to be limited in variety, leading to a restricted concept image being developed from too few exemplars. For instance, the classical differential triangle is usually drawn as in figure 1, with the increments δx, δy both positive and the graph sitting neatly in the first quadrant. As Orton has observed,

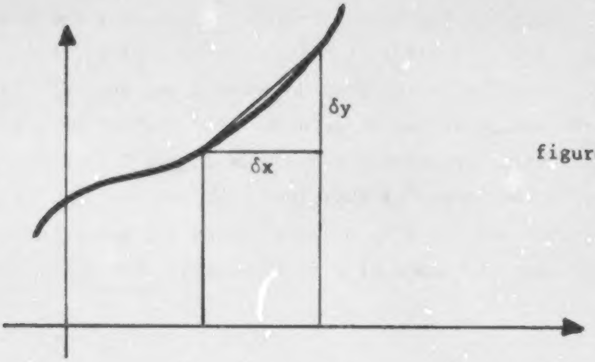

figure 1

a significant proportion of his students interpreted the symbolism

$$\delta y/\delta x \to dy/dx$$

to mean "δy/δx gets smaller until it becomes dy/dx". In an example such as figure 1, the gradient δy/δx *does* get smaller until, to all intents and purposes, it is indistinguishable from dy/dx. By simplifying the examples presented to students in this way, hoping to help them in the initial stages, the net result may be a restricted concept image in the student's mind which later conflicts with the formal theory (Tall & Vinner [6]).

THE COMPUTER PROGRAMS

To combat the conceptual limitations of the kind just described, a suite of three computer programs were written by the first-named author to help generate more appropriate concept images: GRADIENT, AREA and BLANCMANGE. GRADIENT draws moving pictures of the gradient of a graph, leading to ideas of differentiation, BLANCMANGE draws an everywhere continuous, nowhere differentiable function [5] to prevent too limited a set of exemplars being encountered, and AREA computes and displays the area under a graph in various ways, leading to ideas in integration.

GRADIENT and AREA both allow the input of a function in normal analytic notation (e.g. $f(x)=\sin 2x$ or $f(x)=(x^2-1)^{1/2}$) and draw the graph over a chosen range, indicated places where the function becomes undefined or has an asymptote. GRADIENT offers two main routines, the first simulating the limiting process at a point in which the chord is drawn between two chosen points $(a,f(a))$, $(b,f(b))$ and then b moves in steps to a as the gradient is displayed numerically on the screen. On one computer (the 380Z) arithmetic accuracy is such that the gradient can only be obtained to about three figures, seriously prejudicing the concept image of the limiting process, but on another (the BBC) five digit accuracy allows a much more successful simulation. Both computers are markedly better in the second routine, displaying the gradient as a *function* g(x)=(f(x+c)-f(x))/c, for fixed (non-zero) c and variable x. To simulate this dynamically the program draws a sequence of chords from x to x+c as x increases by steps, simultaneously plotting the gradient of the chord as a point (x,g(x)). The static picture in figure 2 (a computer printout) fails to convey the impact of this idea, but the moving picture on the screen, with the function and the step c specified by the user, leaves an unforgettable impression so that the graph of g(x) is visibly seen to be the gradient of the graph f(x).

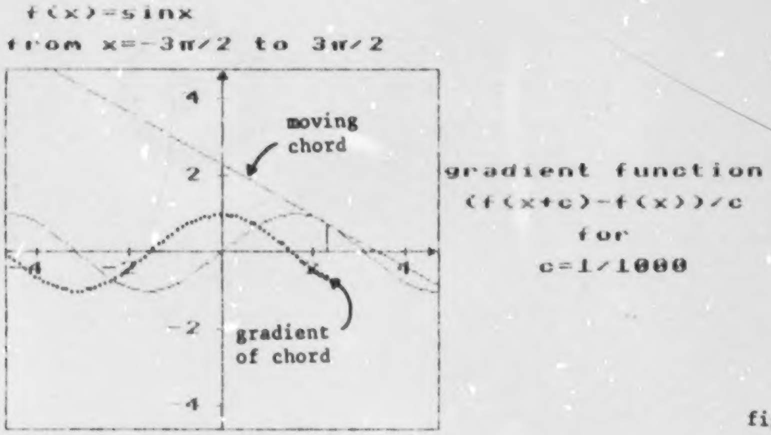

figure 2

As one student put it, "I never understood what it meant to say that the derivative of sinx was cosx until I saw it grow on the computer".

The program AREA also has two major routines, one to display the approximate area under the graph as computed by various methods, the other to draw the "area-so-far function" from a to x as a function of x. Positive and negative areas are displayed in different colours and the reasons for the signs become more obvious when it is realized that the area is calculated as the product of two signed lengths drawn on the screen. For instance, calculating the area from right to left has *negative* step times the signed ordinate, giving a negative area above the axis and positive area below.

The programs are designed both for demonstration purposes and for student investigations, allowing students the freedom to explore and enrich their concept images in a more personal way. It is interesting to see the students regarding the computer as an authority which does not present the same threat to them as the teacher. Indeed, they seem far more willing to discuss conceptual difficulties thrown up by the computer than they would difficulties in understanding a teacher's explanation.

RESEARCH

The second-named author has initiated two research studies using the programs. A cross-sectional study is being conducted on a group of about 30 A-level (=senior high school) students. The group is being divided so that some use the GRADIENT program, some the AREA program and some both. A questionnaire is being administered to all the students to assess their understanding on about 30 different concepts, including a number pinpointed by the Orton study. The data is being analysed in an attempt to identify those concepts that appear affected by the activities. It is hoped to present the results available at the PME conference.

A longitudinal study is also being conducted on a group of twelve adults attending a one year, two-evenings-a-week class designed to take them to degree standard in mathematics. The continuous assessment and teaching style of the small group discussion make this expecially suitable for a study that relies on interpreting written work and contributions to the class. They have recently begun work on the calculus and used the programs in groups of about six students each. Though the full analysis must await the end of the course, preliminary impressions show some interesting reactions.

In the first session the students used the computer to study the gradient of the graph of sinx at a number of individual points. Initially they were invited to choose points a,b quite far apart and to see that as the step between the points decreased the gradients of the chords formed no obvious pattern. They readily appreciated that the step had to be relatively small before the sequence of values for the gradient converged. There was interest and scepticism when for very small steps the gradient began to wander again after having appeared to converge. (This was the 380Z computer with limited accuracy.) They investigated positive and negative steps and one group became interested in the number of stages it took before the gradient stabilized, concluding that it depended on the curvature of the graph at the point.

In the next session the students were introduced to the gradient function, drawing sinx, cosx, then powers of x, and guessing the formula for the derived function which was drawn and compared with the gradient graph. They were very familiar with the graphs of standard functions and correctly conjectured the derivatives in every case. Exponential functions provided the first instance of student generated work. The graph of 2^x was drawn on the screen and the gradient function plotted. The two curves were clearly related and it was discovered by trial and error that the derived function was about $\frac{2}{3}2^x$, a quite reasonable approximation. The graph of 3^x was similarly examined and then e^x. It became apparent that e was the number which, when raised to the power x and differentiated, had a derivative which equalled the original function. Further student investigations initiated by the group led to various other conjectures, the *pièce de résistance* being the conjecture that the derivative of arctan(x) was $1/(1+x^2)$!

The first session devoted to integration was noticeable for the large amount of discussion around the idea of negative area. With many other groups this has created no problems, but this time, principally among students who were primary teachers, there was some resistance to accepting the idea of negative area at all. The program was invaluable in that it could focus the discussion on a picture where the students could see why the area of a strip came out negative in a variety of cases, and that integrating from a higher limit to a lower one gave the same answer as the other way round, but with a change in sign. One group using the program decided to find the "paintable area" between the curve and the axis by dividing the calculations into segments above and below the axis and taking the absolute value before adding. It was then realized that the whole calculation could be done in one go by taking the original function, squaring, then square-rooting before calculating the area. The second group came to similar conclusions but used the abs function instead.

The drawing of the "area-so-far function" from various arbitrary points has naturally given a meaning to the constant of integration and highlighted once more the importance of the change in sign when integrating from high to low rather than vice versa.

It is too early to say what effect the conputer has had on the students' responses to assessment questions, however, it is already noticeable that graph-oriented questions submitted so far have been very well done. More details should emerge when the study is considered as a whole.

Most student reaction has been positive. One student remarked "It was helpful, fantastic, just being able to draw the graphs ... it would have been such a hassle any other way." Another said after the very first session, "It's interesting that it's only looking at the graph that it's made any sense. You know I said I didn't understand what the derived function was all about - I could do all the odd things before but until now I didn't have a clue what it meant."

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G. METHODOLOGY

STRATEGY GAMES, WINNING STRATEGIES AND NON-VERBAL COMMUNICATION DEVICES (at the age of 8)

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Nous avons présenté antérieurement les résultats obtenus, à l'aide de Dispositifs Non-Verbaux de Communication, par des enfants âgés de 6 ans (PME 5), puis ceux obtenus par les mêmes enfants un an plus tard (PME 6). Nous décrivons ici comment certains de ces enfants ont abordé à l'âge de 8 ans des jeux de stratégie et coment ils ont découvert certaines stratégies gagnantes.

BASIS OF THE TECHNIQUE

We call NVCD any non-verbal device which can be used in such a way that it is non ambiguous, simple, flexible ; it should be used in the frame of games and it must have built-in technical constraints which suggest a logical structure. The use of NVCDs seems to favour the development of language and thought in children.

This year, we used NVCDs to study 8-year olds' reactions to strategy games.

MATERIAL AND TECHNIQUE

We worked with 16 children in third grade (8-year olds). 14 of these children had used NVCDs since the age of 6 (LOWENTHAL, F. & MARCQ, J., 1981, 1982a, 1982b).

For the sake of clarity we chose to describe in this paper only the first game and the first device we used. The game is and extension of the NIM-game, the device consists of magnetic pawns put on lines drawn on the blackboard.

We introduced the rules of the game to the children in the following way : "Two players, Rose and Marc, see some pawns on horizontal lines drawn on the blackboard. They play alternately but Rose is always the first to play. Each player takes as many pawns as he wants, provided that they are all on the same horizontal line ; he must take at least one pawn. The player who takes the last pawn loses".

We let the children play by pairs on the blackboard with actual pawns, the rest of the class could make suggestions or discuss the value of the moves.

In order to help them to discover the regularities and strategies we asked for a technique "to help the teacher keep track of what was happening". There were 10 lessons of 90 minutes each.

RESULTS

The children learned very quickly how to play this game. They created a notation system during the first session, using n-tuples as shown in figure 1. At this stage an n-tuple represented for the children the number of lines at the start of the game, the number of pawns on each line at the moment the n-tuple was written down, and the respective position of these lines : (2, 3) meant "2 pawns on the first line and three on the second"; this was considered as totally different from (3, 2). A "0" was used to indicate that a line was empty. During the 7th session, after long discussions, the children agreed to skip the zeros and use only the relevant information (empty lines have no relevance on all further possible moves) ; they also said that, although (3, 5) is not (5, 3), both define the same game-situation.

figure 1a : situation
coded by (2, 3)

figure 1b : situation
coded by (3, 1)

Since the 2nd session the pupils used arrows to connect the n-tuples in order to describe the succession of moves during a given game.

During the 3rd session, one pupils described what the situation coded by (1, 0) meant : "The player who has to play loses", he also gave the correct winning strategy for (n , 0) games, with $n > 1$. 4 other pupils understood this. They started to study winning situations and were able to make short term predictions.

The importance of the situation coded by (2, 2) was discovered during the 4th session. This situation is important because 1°) the second player has a winning strategy , and 2°) all the strategies for 2-lines games can be reduced to this one. This was explicitly used and verbally formulated during the 5th session, when one pupil gave the general strategy for games of the form (2 , n) with $n \geqslant 2$ and another pupil, for games of the form (1 , n) with $n \geqslant 1$. 3-lines games were introduced as homework.

During the 6th session, 4 pupils solved correctly the game (1, 2, 3) and the game (2, 2, 2) by trying to come back to (2, 2) or to (1, 0). They discovered and explained to the other pupils the importance of the situation coded by (1, 1, 1) : "the player who must play is sure to lose".

Finally, during the 9th session, 4 pupils showed that they were able to analyze all possible moves for 2- and 3-lines games, and to use their analysis to state that for (n, n) games, the second player has a winning stratefy, while for (n, m) games, with $m > n > 1$, the first player has a winning strategy (although they did not use exactly the same words as we do !). During the 10th session, these pupils explained verbally what is the strategy for the 2-lines games, using the following words : "If they are the same, you do the same (*as the first player*), but if they are different you make it the same". All pupils accepted these "cooking recipe" and were able to play adequately, but we do not know whether all of them understood why they had to play this way.

DISCUSSION

Horizontal lines were drawn on the blackboard and pawns put on them to represent the different situations : this is a weak form of NVCD (there are no mechanical internal constraints), but our pupils were used to concretely manipulate NVCDs and use these manipulations to introduce representations. They reacted positively to this new device which required manipulations of representations. We do not know what untrained children would have done. According to their teacher, these children, who worked previously during two years with NVCDs, seem to do better, as far as discussion and reasoning are concerned, than children who had studied since first grade with the same teachers as our pupils but had not been exposed to NVCDs.

The device described here, and the one we used during the second part of the schoolyear , enabled us to let children discover that in certain games there is a winning strategy for one of the players, but not always for the same player : this depends upon the starting situation chosen by these players . But, anyway, 8 pupils (out of 16) are convinced that it is always possible to compute in advance who has a winning strategy , once the situation is given. These 8 pupils were able to discover winning strategies in particular cases, but only 4 of them were able to synthetize all the informations discovered by different pupils about different situations, to generalize all this and formulate the correct strategy for the general case. These 4 pupils were even able to formulate their strategy in a verbal and practical way which made it understandable for the 4 other "bright" pupils and usable by all the 16 pupils.

The use of this technique enabled some pupils (8) to discover a need for some kind of reasoning by induction : "We must go back to simpler and already fully explored situations".

The pupils used notations which had been suggested by one of them and not by us. This use of n-tuples corresponds to a notation system we had introduced when the children were 6 years old (in the framework of geoboards) : we did not use this notation system while they were in second grade (7 years old) although some tried to use it then.

Obviourly, the presence of semi-concrete material (real pawns which had to be physically removed in order to play) helped the children to visualise the situation, and later to express verbally whatever they had to say about it. This seems to confirm some of our previous observations (LOWENTHAL, F. & SEVERS, R., 1979, 1980).

CONCLUSION

The results summarized in this paper tend to support the hypothesis we formulated about NVCDs : "The use of a Non-Verbal Communication Device, firstly in a purely concrete fashion, later in a more symbolic way, can favour the development of language and thought in children ; the introduction of an NVCD helps a child to structure his perceptive field (by helping him to process data)". In doing so, an NVCD initiates a complex cognitive process which could not have starded earlier because of problems caused by the processing of complex data : the logical structure suggested by an NVCD helps a child to sort, and thus to simplify complex data. On the other hand, the complexity of the task involved does not seem to cause similar problems : although our pupils do not always say what should be done in the general case, they move their pawns adequately.

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CHILDREN'S LEARNING STRATEGIES IN MATHEMATICS: A PRODUCTION RULE ANALYSIS

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Given a degree of autonomy when learning a mathematical topic, what strategies do children employ? Can they organise themselves to learn effectively? What developmental trends are there? These were the questions that prompted the research outlined in this paper.

The method chosen to explore these questions was developed to investigate adult learning strategies by Gordon Pask (Pask, 1976). The would-be learner is provided with a set of teaching materials, each element of which deals with a particular aspect of the topic involved. These materials could take the form of work cards, computer-administered tutorials, or any other appropriate style of presentation. Typically the set of materials embodies a great deal of redundancy, so that no learner needs to study each element, though he is free to do so if he wishes. The relationships between the elements are made clear to the learner, for he is provided with an 'entailment structure', a kind of concept map, implicit in which is a multiplicity of routes towards the overall goal of mastery of the topic under consideration. The route an individual selects is an overt expression of his learning strategy with regard to that topic.

THE STORY PROBLEM MATERIALS

The children involved in this study ranged from nine to twelve years old, so an appropriate topic for them was story problems, such as:

> Susan went shopping after Christmas to spend some of the money she had been given. She had £5.23 in her purse when she started, but then bought a book costing £2.99 and some felt-tips costing £1.59. How much money did she have left?

The overall goal for the children in this study was therefore to learn how to solve such problems. Successful solution of the example above depends upon two skills. One is the ability to decide what calculation is the appropriate one to perform, and the other is the ability to execute it. An entailment structure representing this initial analysis is shown in Figure 1 If this were as far as it was necessary to go, the only further preparation

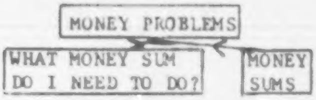

Fig 1: A simple entailment structure

required would have been to create three sets of learning materials, one for each node in the structure. Of course it was necessary to go a great deal further, but there is no space here to describe the gradual process of elaboration that occurred. Suffice it to say that the final result was an entailment structure with 42 nodes, which is shown in Figure 2. The nodes are numbered for ease of reference, the three nodes shown in figure 1 featuring as numbers 38, 33 and 28. The other difference between figure 1 and figure 2 is that the relationships between the nodes in figure 1 are indicated by arrows, whereas those in figure 2 are expressed in terms of inputs and outputs for each node, purely in the interests of clarity. The inputs represent the pre-requisite knowledge required for a given node, and the outputs are a shorthand for the knowledge deemed to have been acquired when the teaching materials for that node have been mastered. For example, node 38, MONEY PROBLEMS, has as its inputs M D MS. M is a shorthand for MONEY and its MEASUREMENT. D is a shorthand for DECIMAL ARITHMETIC. MS is a shorthand for WHAT MONEY SUM DO I NEED TO DO? If a learner knows about pounds and pence (M), can do calculations with decimals (D), and can identify what calculation is the appropriate one to perform in money problems (MS), there is a reasonable expectation that he will be able to put all those together and successfully master node 38. If he actually does study the teaching material for node 38, he will have demonstrated the ability to solve at least some story problems. The output Ⓟ from that node is a shorthand for this. Outputs from one node are inputs for others and in this way the pre-requisite relationships between the nodes are made explicit. There are analogical relationships toc For example, D is the output from five distinct nodes - numbers 28 -32 inclusive. Thus in this structure there are five analogous ways to acquire a facility with decimal arithmetic, differing only in that one involves pounds and pence, another metres and centimetres, and so on.

The teaching material for each node was a computer-administered tutorial written in the Open University's CICERO author language and implemented on its DEC-20. A terminal was installed in the primary school involved, for the nine month period of the study. Each of the 42 children who took part had his own copy of the entailment structure, which was called a chart. Before a child began his work the chart was explained to him and he was

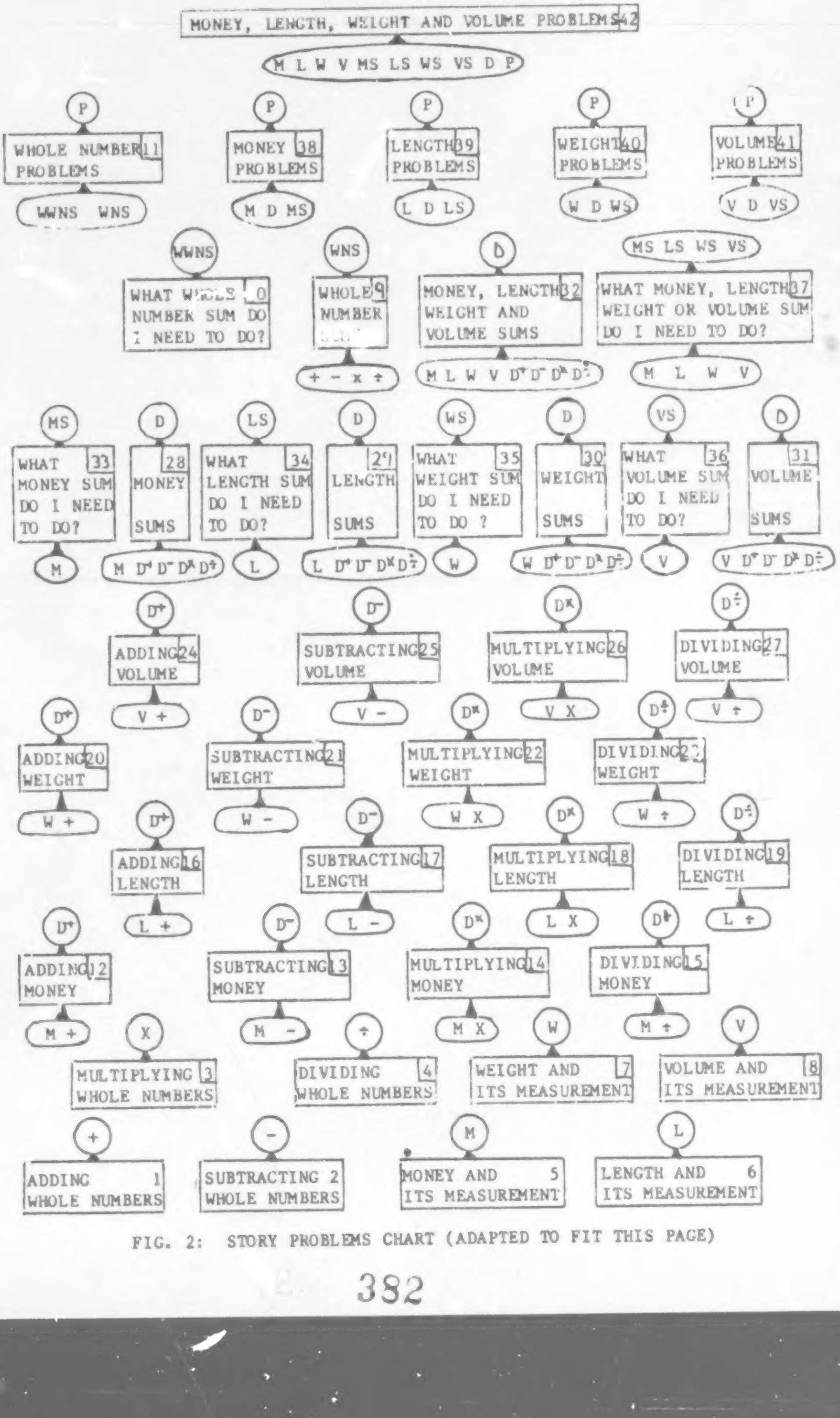
MONEY, LENGTH, WEIGHT AND VOLUME PROBLEMS 42
M L W V MS LS WS VS D P
P
WHOLE NUMBER PROBLEMS 11
WWNS WNS
P
MONEY PROBLEMS 38
M D MS
P
LENGTH PROBLEMS 39
L D LS
P
WEIGHT PROBLEMS 40
W D WS
P
VOLUME PROBLEMS 41
V D VS
WWNS
WHAT WHOLE NUMBER SUM DO I NEED TO DO? 10
WNS
WHOLE NUMBER SUMS 9
+ - x ÷
D
MONEY, LENGTH WEIGHT AND VOLUME SUMS 32
M L W V D+ D- Dx D÷
MS LS WS VS
WHAT MONEY, LENGTH WEIGHT OR VOLUME SUM DO I NEED TO DO? 37
M L W V
MS
WHAT MONEY SUM DO I NEED TO DO? 33
M
D
MONEY SUMS 28
M D+ D- Dx D÷
LS
WHAT LENGTH SUM DO I NEED TO DO? 34
L
D
LENGTH SUMS 29
L D+ D- Dx D÷
WS
WHAT WEIGHT SUM DO I NEED TO DO ? 35
W
D
WEIGHT SUMS 30
W D+ D- Dx D÷
VS
WHAT VOLUME SUM DO I NEED TO DO? 36
V
D
VOLUME SUMS 31
V D+ D- Dx D÷
D+
ADDING VOLUME 24
V +
D-
SUBTRACTING VOLUME 25
V -
Dx
MULTIPLYING VOLUME 26
V X
D÷
DIVIDING VOLUME 27
V ÷
D+
ADDING WEIGHT 20
W +
D-
SUBTRACTING WEIGHT 21
W -
Dx
MULTIPLYING WEIGHT 22
W X
D÷
DIVIDING WEIGHT 23
W ÷
D+
ADDING LENGTH 16
L +
D-
SUBTRACTING LENGTH 17
L -
Dx
MULTIPLYING LENGTH 18
L X
D÷
DIVIDING LENGTH 19
L ÷
D+
ADDING MONEY 12
M +
D-
SUBTRACTING MONEY 13
M -
Dx
MULTIPLYING MONEY 14
M X
D÷
DIVIDING MONEY 15
M ÷
X
MULTIPLYING WHOLE NUMBERS 3
÷
DIVIDING WHOLE NUMBERS 4
W
WEIGHT AND ITS MEASUREMENT 7
V
VOLUME AND ITS MEASUREMENT 8
+
ADDING WHOLE NUMBERS 1
-
SUBTRACTING WHOLE NUMBERS 2
M
MONEY AND ITS MEASUREMENT 5
L
LENGTH AND ITS MEASUREMENT 6

FIG. 2: STORY PROBLEMS CHART (ADAPTED TO FIT THIS PAGE)

provided with a brief description of each tutorial, which he could consult to help him make his decisions, if he wished. Each child recorded his progress and his current plans on his chart. Any reasons he gave for his decisions were noted. The next section consists of summaries of the protocols of three of the children, to illustrate the kind of data obtained.

THREE PROTOCOLS

Paul: began with tutorial 8, VOLUME AND ITS MEASUREMENT, because it had no prerequisites (inputs), but failed. Then chose another tutorial with no inputs, LENGTH AND ITS MEASUREMENT, and succeeded. Scanned the chart and decided he knew about money and adding, so tutorial 12, ADDING MONEY, should be manageable. He therefore attempted it, and succeeded. Commented that he now thought he should be able to do the other adding ones, and attempted tutorial 16, ADDING LENGTH, successfully. Scanned the chart again and decided to do tutorial 9, WHAT WHOLE NUMBER SUM DO I NEED TO DO?, again because it had no inputs. Commented that he now thought he should be able to do 'the other ones like that' and picked on tutorial 33, WHAT MONEY SUM DO I NEED TO DO? Failed at this one, but repeated it immediately, this time with success. Continued with another analogous tutorial, number 35, WHAT WEIGHT SUM DO I NEED TO DO? and succeeded. Scanned the chart again and picked on tutorial 4, DIVIDING WHOLE NUMBERS, because it had no inputs. He failed on his first attempt, but repeated it immediately, and succeeded. Scanned the chart again, settling on another adding tutorial, number 20, ADDING WEIGHT, and succeeded. There was no more time available so he had to stop there.

Christopher: Began by scanning the chart, looking for something he thought he ought be able to do. Decided that all the single operation arithmetic tutorials should be manageable. Picked tutorial 24, ADDING VOLUME at random from these and succeeded. Next he chose tutorial 25, SUBTRACTING VOLUME, for similar reasons, and succeeded. Observed he'd now acquired a D^{+} and a D^{-}. Said 'I need to do a multiply and a divide'. Attempted tutorial 19, MULTIPLYING LENGTH, and then tutorial 20, DIVIDING LENGTH, succeeding with both. Noticed that he now had all the inputs for tutorial 29, and that it would yield him an output (D) he hadn't yet obtained. Attempted tutorial 29, LENGTH SUMS, and succeeded. Scanned the chart again, looking for something manageable which would also yield him something new. Eventually chose 32, MONEY LENGTH WEIGHT AND VOLUME SUMS, because of its similarity to the one he'd just done and the chance it provided to do something with money and weight. This was successful. Scanned the chart again, still looking for something manageable and which would yield him something new, and noticed that tutorial 37, WHAT MONEY, LENGTH, WEIGHT AND VOLUME SUM DO I NEED TO DO? would provide 4 new outputs at once. Attempted it successfully. In a similar way he decided he should then do one of the problems tutorials. Chose tutorial 38, MONEY PROBLEMS and succeeded. Now the only tutorial that would yield something new was the final one, tutorial 42, MONEY, LENGTH, WEIGHT AND VOLUME PROBLEMS. Successful with this, so 'completing' the chart.

Joanne: 'I'll try one of the bottom ones first. They should be easier than the ones higher up'. Attempted tutorial 8, VOLUME AND ITS MEASUREMENT, and succeeded. 'That was easy, I don't need to do any more of those (5,6,7) I know I can do all those (1,2,3,4). I think I should be able to do all these ones (12-27) but I'll try one to see'. Attempted tutorial 24, ADDING VOLUME and succeeded. Scanned the chart and focussed on the WHAT SUM DO I NEED TO DO? tutorials. 'I wonder what those are like?' Attempted tutorial 33, WHAT MONEY SUM DO I NEED TO DO? and succeeded. 'That was fun. I'll do another like that'. Scanned the chart to identify the others and 'I can get lots of things at once if I do that one!' (tutorial 37, WHAT MONEY, LENGTH, WEIGHT AND VOLUME SUM DO I NEED TO DO?) Completed it successfully. Scanned chart to see what she could do now. Dismissed 34-36 because none would give her any new output. Noticed how near she was to having all the prerequisites for the top tutorial. 'I only need a D and a P'. Looked to see where D could be obtained. 'I could do any of those (28-32) I'll do 28'. Attempted tutorial 28, MONEY SUMS, and succeeds. 'Now I only need a P'. Looked to see where P could be obtained and identified the five possibilities (11, 38-41). Attempted tutorial 38, MONEY PROBLEMS and failed. Didn't want to repeat it, preferring to try another of the possibilities. Attempts tutorial 39, LENGTH PROBLEMS, and succeeded. 'Now I can do the top one, I've got all the things'. Attempted tutorial 42, MONEY, LENGTH, WEIGHT AND VOLUME PROBLEMS and succeeded, so 'completing' the chart.

PROTOCOL ANALYSIS

These protocols are being analysed within an information processing model known as a 'production system'. A production-system model has three fundamental components. The first component is a working memory, which contains a limited number of pieces of information, in the order which they were added to the memory. In this study elements regularly found in working memory would be SUCCESS and FAILURE, referring to the most recent tutorial attempted. The second, and principal component is a set of production rules, which are all statements of the form

IF conditions A, B... obtain, THEN actions X,Y...are appropriate.

An example relevant to the protocols in the previous section would be

IF a new tutorial is needed, THEN search the chart for something achievable.

Very often, more than one rule is applicable at any one time, so there has to be a mechanism for deciding which one to implement. This is not a trivial problem, for implementing any rule may change the conditions obtaining, with the result that a rule passed over at one point may never be applicable again. Thus the third important component of a production system is a set of criteria for assigning priorities to rules. These are usually known as conflict resolution principles and the ones used in this study are those suggested in McDermott and Forgy (1978). These are:

(a) refractoriness: no rule is ever fired (implemented) more than once in response to the same set of conditions. (This prevents the system being caught in a closed loop).

(b) recency in working memory: the more recent the addition to working memory of its conditions, the higher the priority of a rule.

(c) special case: if the conditions for one rule are a subset of the conditions for another rule, the first rule is removed from the set of currently applicable rules.

(d) if there is still a conflict, the newest rule is used: if there is more than one such the rule to apply (fire) is chosen at random.

Production systems of this kind have been used to model human problem solving (Newell and Simon, 1972) and the Piagetian tasks of seriation (Young, 1976), and number conservation (Klahr and Wallace, 1976). They have also been used to model subtraction algorithms (O'Shea and Young, 1978) and are currently being used to model physics problem solving and fraction algorithms at the Open University. They are also the stuff of which 'expert systems' are made, and are a key tool in artificial intelligence work. Gradual improvements in performance on tasks are readily modelled by adding further rules to an existing set, by modifying those already present, or sometimes deleting them completely.

I have used this preliminary paper to provide the background for the application of production-rule analysis to the protocols produced by the children working with the story problem chart. In my presentation I shall give actual examples of sets of production rules which produce behaviours like those shown in the protocols.

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A NEW APPROACH TO MATHEMATICS TESTING

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INTRODUCTORY NOTE

This paper provides the setting for a new research project on alternative forms of mathematics assessment in Secondary Schools. In the session at PME I will present and discuss our continuing attempts to devise valid and reliable mathematics tests for a national system of examinations, in the areas of problem solving, investigations and group work.

BACKGROUND

In general, British secondary schools are organised, in their later years at least, to lead pupils towards national public examinations at age 16+, following five years of secondary schooling. There are well-documented reasons why and how this position has been reached, but for a number of years there has been a growing feeling that an alternative approach to assessment in our schools is needed. This concern arises from the premiss that the present system is inadequate to cope with both the academic needs of the majority of pupils and their personal development. For most pupils the record of their entire secondary schooling is based on their degree of failure in one formal examination which is seen as largely irrelevant to their futures.

It is in this context that an educational certificate, designed to be markedly more free from the perceived constraints referred to, has been proposed. The ideas had been around for some time but were highlighted by the continuing indecision over the proposed new system of examining at 16+. The provision of alternative written examinations at 16+ may be seen only to preserve the existing problems of educating pupils in schools and to do little to provide incentives for pupils, particularly the less able.

How can pupils be better motivated and helped to gain more from school in a society clouded by unemployment?

O.C.E.A.

The Oxford Certificate of Educational Achievement (O.C.E.A.) has been proposed as a means of tackling some of these problems, and the aim is to provide information which matches the needs of schools and the employment market in its fullest sense. (Higher Education, Further Education, Industry, etc.)

The Certificate will be divided into three parts: a personal profile, records of the graded levels of assessment attained, and details of existing qualifications. The intention is that there be no formal link between the Certificate and any particular year of schooling. It is with the second of these items - the assessments levels - that I am concerned in this paper. At present four broad areas of the curriculum have been designated for testing: English, Mathematics, Science and Modern Languages.

Following a review of the implications for the mathematics curriculum, a system of "Graded Objectives Assessment Levels", or GOALS will be devised, which will allow pupils' performances in areas of the secondary school curriculum to be assessed in a number of finite steps. Fundamental to the thinking on OCEA is the notion, endorsed by the Cockcroft Report(a), that children should be judged by their successes, not their failures. Each GOAL test will either be passed or not. The individual tests will not be graded. Tests at each of the various GOALS will be made available two or three times a year, and pupils will be able to attempt each GOAL as and when judged to be ready to do so by the teacher in the school. The assessment levels will be designed to cater for the whole ability range and the intention is that pupils would not be entered for a particular GOAL test without a good expectation of success.

By breaking down the curriculum into GOALS, pupils' motivation may be maintained by the provision of discrete steps of success to be achieved and built upon before the end of the fifth year when most pupils now take public examinations. The Certificate is thus intended to provide a useful stimulus to pupils of all abilities.

THOSE CONCERNED

The work is being undertaken by a partnership of four local Education Authorities, Oxford University Department of Educational Studies and the Oxford Public Examination Board. To be of value in the outside world the

Certificate must obviously have support and validation from a public examination body, but it also has to reflect what is needed in schools, and for this reason teachers form a large proportion of those concerned in its development. Once the GOAL tests and personal report element become available, in-service training for those concerned with the Certificate, both from the production side (the mathematics teachers) and from the interpretation side (employers) will be needed.

THE IMPLICATIONS FOR MATHEMATICS TEACHING

Examinations undeniably have a profound, some might say overwhelming, influence on the curriculum. Those of us involved with OCEA therefore feel that this is the moment to re-think the approach to the teaching of mathematics. Two fields of influence are open to us: the curriculum and the methods of assessment.

The approach taken to the construction of the various levels is that of "bottom up" rather than "top down". That is to say that a basic core of essential mathematical attainment will initially be defined and succeeding levels built up appropriately on this foundation. The basic core will take account of Cockcroft's foundation list(b) and the expectation is that it will be mastered by 90% of all pupils by the time they leave school. The system at present in existence is more likely to engender a school syllabus based on O-level requirements and watered down for younger or less able pupils.

It is, however, probably in the area of assessment that most can be done to encourage the good practices suggested by the Cockcroft Report(a). If we want mathematical activities such as group work, investigations, problem solving and discussion to go on in the classroom then its value must be seen to be reflected in the examination system. The traditional paper and pencil test will no longer be appropriate as the *sole* means of testing mathematical understanding and achievement.

New attitudes to pupils working together at an assessable project must be fostered. Reliable methods of testing problem solving skills must be designed. We must create probes to expose the real thinking of a child working on an investigation.

It is in the above mentioned areas that the mathematics steering group is now working, and I shall appreciate any contributions which members can offer to this largely unexplored field of research.

The skeleton of OCEA has been constructed, but much research is needed before the bones can be covered in flesh.

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A TASK-ORIENTED METHOD OF PROTOCOL ANALYSIS*

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1. INTRODUCTION

The technique of protocol analysis is, in fact, a combination of analysis and synthesis. As the protocol usually exhibits a partial and idiosyncratic record of the solver's solution process, part of the researcher's analytic task is to remove the idiosyncratic flavour by describing the process in alternative language. Furthermore, the researcher tries to fill the omissions in the protocol by making inferences in the contexts in which gaps occur. However, the main task in protocol analysis is to make some pertinent generalisations about problem solving behaviour of subjects across a set of experimental problems, which means that the protocol must be analysed according to some "synthetic" categories.

There are several variables which may influence the choice of such categories and which, in turn, limit the applicability of a particular model of problem solving in another experimental setting. Among these variables are included:

i) Choice of tasks - either requiring substantial domain-specific knowledge ("semantically rich domains") or little (most of the so-called "non-standard" problems)

ii) Choice of subjects - varying both in the extent and kind of problem solving training as well as the degree of domain-knowledge expertise that they posess.

iii) Choice of methodology - even when the "think aloud" technique is the main source of data collection, there are many variations within the technique. These include; intial instruction to the subjects, degree of interviewer interventions, individual or group solution effort.

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*The research was jointly undertaken by the author and Professor D. Wheeler. It was funded by the Quebec Government, F.C.A.C Grant EQ. 1261

We elaborate on the above points by briefly discussing two models of problem solving, one of Information Processing and one suggested by Schoenfeld. We then comment on the applicability of these models to our own experimental situation.

Information Processing (I.P.) psychologists generally model the behaviour of subjects, both "naive" and "experts", solving "well structured" problems such as the Tower of Hanoi (i.e. problems in which the intial state, goal state and the admissible operations are explicit). Since the solver's main task is to construct a goal oriented sequence of admissible operations, I.P. models emphasize the solver's solution process ("solution path") within a given "solution space". Thus the models favour strategic categories such as "selection of subgoals" and "means-end analysis" to account for the solver's behaviour (see Simon, 1978).

Schoenfeld (1982) considers the solution of geometrical problems by college students. He argues that a protocol analysis must be based on three general categories which he calls "tactical" knowledge (e.g. facts, procedures and domain-specific knowledge), "control" knowledge (e.g. "managerial" behaviour and self-monitoring) and "belief systems" of the solver (about mathematics, about the experimental setting and about himself). Schoenfeld suggests that each of the above categories may dominate in a particular experimental setting. For example, a protocol analysis based on the solver's beliefs about mathematics may, in some cases, provide the best explanation of his solution behaviour.

In our research, the subjects ("naive" solvers, age 13-15) were given "non-standard" tasks (both in the sense of being unfamiliar and of requiring little domain-specific knowledge) which were not "well structured" in the I.P. sense. Neither of the above models fitted very well the behavioural items of our solvers. We saw little evidence of means-end analysis since the experimental tasks were not always amenable to the selection of subgoals nor was the goal state sufficiently known in advance. (In spite of the more elastic interpretation of means-end analysis offered by Simon (1978), we feel that his suggestion (1979) that it is an invariant of human cognition must be substantially qualified). On the other hand, Schoenfeld's categories, though clearly more comprehensive than those of the I.P. models, seem more appropriate to domains that are "semantically rich". There was little in our subjects' behaviour that we could characterize as "managerial decisions" nor did it seem that their "tactical" behaviour was "strategically" driven, as suggested by Schoenfeld's model.

Both of the above models fail to account for the most striking aspect of the protocols that we analyzed, namely, the existence of a temporal and psychological "lag" between the intial perception of a problem and the point at which the subjects knew what the problem required. In Schoenfeld's case, the subjects were familiar with the tasks while I.P. models "assume that an intial understanding process has previously run to completion " (Simon, 1979). (We should add that Simon (1978) gives a much more flexible model of problem solving in which the "process of understanding" has a prominent part.)

2. DEMANDS OF A TASK

The fact that achieving the necessary clarity about the problem often takes the solver a disproportionate amount of the total solution time is not suprising. The solver must extract from the problem statement all the overt and covert information it contains about the given, the means, and the goal; he must be quite clear about the distinctions between these three categories of information and yet keep a view of the problem "as a whole" in which the three components interact. This is a considerable analytic-synthetic task for a solver in the presence of an entirely unfamiliar problem.

We try to categorize the multiplicity of demands that a problem makes on the solver in the following way:

(a) Mathematical demands These include the mathematical knowledge necessary to understand the problem statement, as well as the "tactical" knowledge required for the solution.

(b) Structural demands These depend on several components, namely:

-relations embedded in the problem statement (both explicit and implicit)

-the nature of goal-definition (either open-ended or explicit, containing the goal state or only a criterion to determined that it has been attained)

-the nature of operations (prescribed or needed to be clarifie and constructed by the solver)

-sequencing of operations (goal may be reached in a single operation or by a sequence of several operations)

(c) Psychological demands The obstacles to an immediate solution may be caused by a tendency of the problem to create a "set" or fixation, to induce a particular "problem representation", or by making some unusual logical or perceptual demand.

(d) General demands These involve factual (non mathematical) knowledge, as well as affectivity of the solver.

We follow by applying the task-analysis to one of the problems:

Square Cutting Problem: Can you cut a given square into n square pieces? (intially n=4,7 . In some cases also n=10, 11)

Mathematical demands Knowledge of meaning of square; how to divide into squares using a uniform grid.

Structural demands

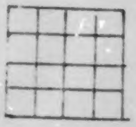
Embedded relations: Relations between cuts and the resulting number of squares; Smaller squares need not be congruent; Operations can be applied to smaller squares.

Nature of goal definition: Goal is explicit; cut a square into a specified number of pieces. The goal state is not given but there is a criterion by which a goal can be recognize when it is reached.

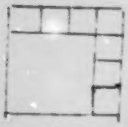
Nature of operations: Vague notion of a "cut" must be clarified as well as the procedure for counting the total number of squares. The solver must then construct different operations such as: grid, bordering, clustering, etc.

Sequencing of operations: n=7, 10, 11 requires a sequence of operations (e.g. grid followed by clustering).

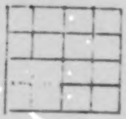
grid:



bordering:



clustering:



Psychological demands: The main obstacle to a solution is the "set" that the smaller squares must all be congruent (possibly because "square" has strong association with "equal", i.e. equal sides, equal angles, equal diagonals). The solver must also distinguish conceptual from perceptual notions of a square in deciding which operations are legitimate.

We note that while the psychological demands are obstacles "created" by the solver because of his perception of a part of the problem situation, both the mathematical and the structural demands are intrinsic to the task. Furthermore, since the mathematical demands of our "non-standard" problems were light, we expect the "weight of difficulty" of the problems to be found among the structural demands. For example, in the above Square Cutting problem, the weight of difficulty lies in the construction and the sequencing of operations. While the weight of each problem resides in different components of the structural demands, it is possible to account for each by stressing one or more of the following stages, viz.

Stage 1: Uncovering the embedded relations contained in the task (R).
Stage 2: Identifying and/or constructing appropriate operations (O).
Stage 3: Sequencing the operations and coordinating them with the problem relations (S).

3. A BRIDGE BETWEEN TASK ANALYSIS AND PROTOCOL ANALYSIS

The three-stage model which we have used to determine where the weight of difficulty of a problem lies- ie. the main demands which a solver must meet - can now be used as a framework for looking at what what solvers actually do. The protocols are analyzed by looking for the different R (relations), O (operations) and S (sequencing) items, and then arranging the items in separate R, O, and S clusters (and chronologically within each cluster).

In carrying out an analysis using the ROS framework, it becomes evident that the solvers verbalize most clearly about the S-items since they correspond to what is percieved to be the most important part of their activity. Thus, S-items are easier to locate and identify than the R- and O-items which, often, need to be inferred.

The separate R, O, and S clusters bring out dramatically the overlaps in time between these categories. There is ample evidence that most solvers do not produce all the R-items, then the O-items and then the S-items in strict logical order. Rather, the occurrence of unsucessful S-items ("trials") is followed by the addition of some R- or O-items (further specification of the problem space).

In summary, we would say that the ROS-analysis emphasizes the solver's construction of the problem space rather than the solver's solution path. A study of the R- and O-items in each protocol appears to give a reliable picture of the solver's developing perceptions of the problem. The positions of these items in the protocol show the steps in the construction and allow us to tell in what "problem space" the subject is working at each moment. The analysis seems to give a meaningful account of the problem solving behaviour of "naive" solvers attempting "non standard" tasks.

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MATHEMATICAL MODELS
A HELPFULL INSTRUMENT FOR EMPIRICAL INVESTIGATIONS ?

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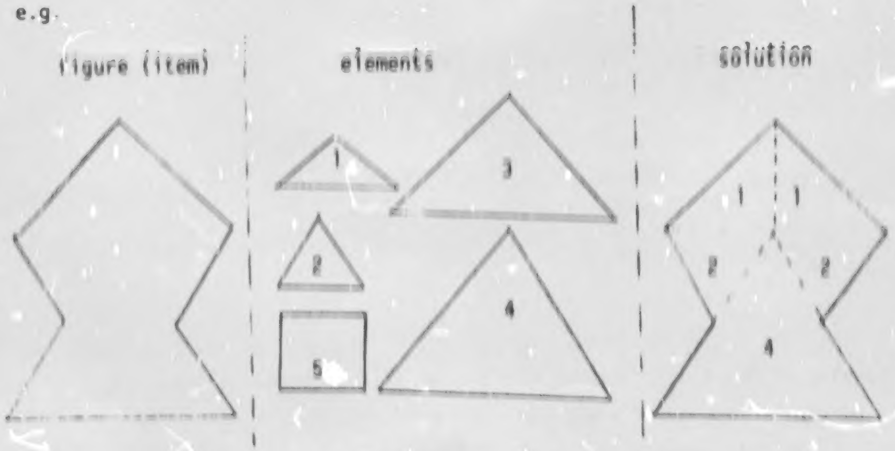
collaborating in a team with U.Lehnert, G.Leßner and W.Reitberger

Most of the investigations concerning effects of transfer of learning don't show convincing results mainly because the problem to define exactly the dimensions of difficulty of the test-items.Are those dimensions to be defined by the mathematical structure of the items - or by psychology operations needed for solving the items?
We tried to find a mathematical model to measure the dimensions of difficulty of the test-items - in quality and quantity.
Non-metric-multidimensional-scaling (NMDS) may help us.We only have to estimate similarities concerning the difficulty of the items.The result of NMDS is a placing of items in a n-dimensional space - similar items near by each other, dissimilar items far of each other.The n dimensions may be interpreted by dimensions of difficulty.
But how to estimate the similarities?
And how to interprete the coordinates of the n-dimensional space?

In order to test the mathematical model "Non-metric-multidimensional-scaling" (NMDS) as an instrument to define different dimensions of difficulty in solving problems of school - mathematics we decided to choose geometrical problems as "tangram" .A differntiated description is given in the abstract of the presentation:Heink, Lehnert, Reitberger "Dimensions of difficulty in solving geometrical problems" published in the "Proceedings of the Sixth International Conference for the Psychology of Mathematical Education".
The task was to cover completely given geometrical figures by elements of the MATEMA - material : special triangles, squares and rectangles, using at each step elements as large as possible.Each step we called an item.

e.g.



We want to descripe the further development of our empirical investigation and discuss some problems.

In the last version we tested 60 pupils with 27 items.

The "Non-metric-multidimensicnal-scaling" based on the estimation of similarities or dissimilarities.
One of our problems was to define such similarities without wasting information .
Our first definition: A pair of items is similar (code:0) if both of the items are solved correctly or incorrectly otherwise it is not similar (code: 1).Our second definition considered the information we had received by registering the mistakes : we didn't consider only correct and incorrect solutions - we discriminated between the mistakes
a) using a scale with the scores :0, 1, 2, 3
b) using a scale with the scores :0, 1, 2, 3, 4, 5, 6
Thus a pair of items was only similar (code: 0) if they were conform concerning the method of solving or making mistakes.
In order to find a matrix of dissimilarities we added the codes of the compared items ($\frac{27 \times 27 - 27}{2}$ pairs) of all pupils.

Another problem : We wanted to find out dimensions of difficulty concerning the items.There was no problem to define a scale for the "global difficulty" of the items.We only had to pay regard to the number of pupils who were unable to solve the items.But can we expect to find the same dimensions of difficulty concerning items very extreme in the "global difficulty" ? Isn't there the possibility that with items of very high "global difficulty" we have to pay attention to other dimensions than with items of very low "global difficulty"?

We therefore decided only compare items for which the "global difficulty" didn't differ too much.

Thus we compared items in special "windows".We first arranged the items with respect to the "global difficulty", then we compared the conformity of the pairs constructed by the first 10 items (1 - 10), then the conformity of the pairs constructed by the items 2 - 11 , etc.

We got final matrix of dissimilarities by considering the arithmetic medium.

We tried this method with "windows" of different width.

For all these different versions we tried EDV - programs of NMDS in order to find a distribution of the items in a space - with dimensions we were able to interprete.After having run programs with one to nine dimensions, we came to assume a solution with two or three dimensions because of the stress.But the interpretation of the dimensions was as difficult as in former versions mentioned in the presentation last year.

The next step was to cluster the pupils (Lemmert).

We found three groups of pupils when companing the mistakes on a scale with 6 scores.We could assume that the probands of each group had tried similar solutions for the items, that they had seen similar aspects in the problems.

The NMDS - program was run for each group separately.

Two dimensions seemed to be essential for the mistakes of all the probands, the other dimensions could be interpreted in different ways for each group.

Then we tried another method (Reitberger).

We reflected once more on the EDV- program of NMDS.The first step for the computer is to use the matrix of dissimilarities of the items in order to make out distances between the items regarding them as points in a space. When using the "method of gradients" by Kruskal there is the danger to find "local minima" dependent on the start.

We therefore tried to help the computer in finding starting-constellation that made sense.At first we devided the pupils in two groups, with low and high "global difficulty".

Within each group we tried to pay regard only to the different ways of solving concerning scores 0, 1; we counted for each pair of items the inconformity of the solutions, neglecting the "global difficulty".

Pbn / Items	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
I_r	1	0	1	0	0	1	0	0	0	1
I_q	1	1	0	1	1	0	1	0	1	0

The inconformity of the solutions of the pair I_r / I_q is : $\frac{8}{10} = 0,8$

I_r has a "global difficulty" : $\frac{4}{10} = 0,4$

I_q has a "global difficulty" : $\frac{6}{10} = 0,6$

The difference concerning "global difficulty" is : 0,2

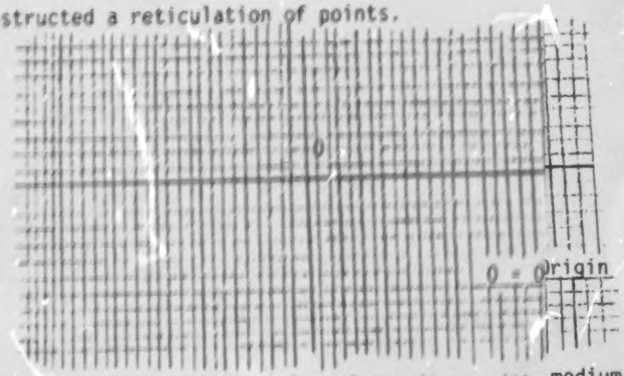
The difference between the types of difficulty is the inconformity of the solutions of the pair minus the difference between the "global difficulties":

$$0,8 - 0,2 = 0,6$$

We only considered the difference between the types of difficulty to find a measure for the matrix of dissimilarities.

We made up our minds to look for a solution in a space with two dimensions.

We constructed a reticulation of points.



We considered the dissimilarities of two items with medium "global difficulty" and tried by special computer program to represent them by two points of the reticulation. In order to place the third item we tried the other points of the reticulation. The program accepted the best fitting, three further fittings were stored. The program then made a translation of the solution to the origin. The fourth item was considered; distances to the three placed items were calculated, the best fitting selected etc. Thus, the program built up a starting - constellation beginning with items of medium "global difficulty" in a reticulation.

This starting - constellation was the input for the normal NMDS -program. The solution was an arrangement of the items in a space with two dimensions. We could interprete the dimensions as follows:

1.dimension : Degree of possibility to see a structure in the figure
2.dimension : Degree of possibility to identify the sides of the given figure (in the subjektive view of the pupil)

These interpretations made sense for most of the items, but for some items they didn't.

For one of these items we compared the suggested coordinates in the first computer-program (the program that placed the items in a reticulation).We selected other coordinates with good fittings stored by the program that made more sense concerning the interpretation of the dimensions and ran the program once more.

The result was a better one.We repeated the same procedure for the other items and became more and more satisfied with the solution by the NMDS.

We are full of optimism that there will be a real chance to use the procedure of NMDS - till now used as an excellent instrument in social-sciences and in marketing - also for effective research in mathematics education.Additional reflections, variations and further development will be necessary to make this instrument applicable for our intention.

The presentation will describe in detail our empirical investigation and our reflections and we hope that there will follow an interesting discussion. For the next year we plan a new empirical investigation.We will try to find dimensions of difficulty for problems concerning the mathematical topic "proportion".

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ON THE CORRECTNESS OF MATHEMATICAL CONTENT IN TEXTS

Joop van Dormolen, Institute for Teacher Training, University of Utrecht

The presentation shall be concerned with a model of characteristics of mathematics that has been developed in order to make the analysis of texts possible. It is part of a research on the possibility of the analysis of learning texts.

Teachers, authors, student teachers, teacher educators, who are professionally engaged in the teaching of mathematics, require a method to analyse learning texts in order to be able to evaluate them in a rational way. By a learning text I mean a relatively small entity in which a certain subject is covered, the entity being a paragraph or a chapter from a textbook, a learning module or any other text to be used in schools for secondary education.

There are three stages in which the analysis can be (and should be) made:

- *a priori*, that is before the text is being used during a learning-teaching process;
- on the basis of observations on the way. The text is being used *during* that process;
- *a posteriori*, i.e. after the process, if one wants to consider in what way the text has contributed to the learning results.

The method that has been developed in my research, refers to the *a priori* analysis in such a way that professionals, like those mentioned above, will find a practical support in it for their decision whether and in what way the text in question can be used in the teaching-learning situation they have in mind.

To this end the method has been split up in groups of focuses, viz.

- *correctness* of the contents;
- *preparation* of the student by earlier and future learning tasks within the curriculum;
- *adaptation* of the text to the abilities of the students.

The focuses are not criteria for evaluation, because *a priori* evaluation depends on the way the analyst thinks how the text is to be used in the classroom and on the weight he intends to attribute to the various focuses.

In describing the characteristics of mathematics, I distinguish two 'dimensions', viz. in the first place that of problem situation and kernels and their relations, and secondly that of aspects of mathematics and their relations. Activities that are focussed on these relationships give mathematics a dynamic nature.

Kernels and problem situations

- *Kernels*

Those parts of mathematics that have traditionally been called "contents" are called kernels. They are mathematical statements like theorems, definitions, rules, axioms, working methods, algorithms and conventional agreements.

- *Problem situation*

Kernels are either the results of activities starting from problem situations, or are being used to solve problem situations. In the former case these problem situations are called starting points, in the latter they are applications. Problem situations can either be of a purely mathematical or of a non-mathematical nature. Kernels can be new problem situations.

Relations

Activities relating problem situations with kernels may have many forms, like generalizing, abstracting, giving concrete form to a generalization, devising analogies, drawing, computing, formalizing, quantifying, ordering, etc.

Aspects of mathematics

- The *theoretical-structural* aspect consists of kernels that are part of a mathematical theory or structure, or of problem situations aiming at discovering or exploring such kernels. Examples of such kernels are a theorem, an axiom, an axiom system, the definitions of a mathematical concept, a set of rules that lay down a mathematical structure.
- The *algorithmic* aspect refers to kernels describing an algorithmical process, or problem situations that aim at finding or applying such a process. Included are drawing procedures like the construction of a line going through a certain point and perpendicular to a certain line.
- The *methodical* aspect concerns non-algorithmic working methods, like heuristics. If one has enough routine, some of these may eventually develop into algorithms.
- The *communicative* aspect consists of conventions about notation, symbols, diagrams and other pictures, modelling schemes for writing down proofs, etc.
- The *logical* aspect consists of kernels describing the formal relations or formal structures of statements, of ways of reasoning, of kinds of proofs, etc., and of problem situations that aim at finding such kernels or applying them.

Relations between aspects

A kernel may have several aspects. A formula for instance may serve as a buildingbrick for a theory, as a description of an algorithmic procedure, as a means of communication and as a problem situation aiming at finding a logical structure. It depends on the context which aspect will be accentuated at a certain moment. Eventually an insight into the several aspects of one kernel will establish the relationship between these aspects.

Dynamical mathematics

In my opinion mathematics should be of a dynamic nature. This can be promoted in education by paying explicit attention to the above-mentioned characteristics, in particular to both kinds of relations. There is no need, however, to reject a text which does not contain all these characteristics. There are two reasons for that.

Firstly, because some of the characteristics may arise in education by other means than through texts. In some cases this is even a must (cf. dilemmas).

Secondly, there may be reasons of a psychological nature why certain kernels cannot be presented as such to or by students, while yet the aspect is implicitly included in the learning activity. This is particularly the case with the logical and methodical aspects, because these often require a relatively high level of cognitive understanding.

In the presentation I want to show the analysis of several parts of learning texts for secondary schools.

H. TEACHERS AND TEACHING

A CASE STUDY OF TEACHER THINKING AND STUDENT DIFFICULTIES

Nurit Zehavi and Maxim Bruckheimer
Weizmann Institute, Rehovot, Israel

Teacher awareness of student cognitive ability determines, to a certain extent teaching strategies. Clark and Yinger (1979) emphasize that what teachers do, is affected by what they think and thus one of the main concerns in research on teaching is teacher judgement. From the small number of studies on judgement (Hook and Rosenshine, 1979) it is clear that teachers vary as to the accuracy of their prediction of student achievement.

In a previous study (Zehavi and Bruckheimer, 1981) a method of analyzing tests using the teacher prediction was introduced. The method provides information about student-teacher-program interaction. Items and topics for which a fairly consistent mismatch occurred, where further investigated for causes (Zehavi and Bruckheimer 1983). Yet for several items, dealing with the application of equation solving to the notion of function, serious *inconsistent* discrepancies were found between teacher expectation and student performance. As a result another study was carried out, in which we tried to "remove" the inconsistencies by looking at different groups of teachers and prospective teachers, and seeing if teacher expectations, in this topic at least, had some connection with teache education and experience. Since teacher expectations are also based upon his view of student difficulties, we included in the study a request to teachers to justify their expectations in the light of possible student difficulties.

The Study

Student population

The Rehovot junior high school mathematics program has been developed for three ability levels. Seven top stream ninth grade classes (n = 225) in "good" schools participated in the study.

Teacher population

Four groups of teachers and student teachers (n = 55) participated. Breadth of teaching experience and type of mathematical education are specified in the following table.

Group	Breadth of teaching experience	mathematical education
high school teachers (n = 15)	7 - 12 grades	university
junior high school teachers (n = 10)	7 - 9 grades	college
university student-teachers (n = 16)	none	university
college student-teachers (n = 14)	none	college

Note: It would have been interesting to include a group of junior high teachers with a mathematics degree, since this would allow a comparison with the group of high school teachers, in which breadth of teaching experience if the major variable. Unfortunately, there are not many such teachers and we could not form a group for this study.

The questionnaire

The students were given the questionnaire, *Functions, calculations and substitutions* consisting of six items. They were asked to write a "plan" for solving each item and a detailed solution.

The four groups of teachers were asked to consider the following three of the six items.

(1) Given that $f(x) = ax^2 - 3$ *and* $f(2) = 29$, *find* $f(6)$.

(2) Given that $f(x) = ax^2 - 15$ *and* $f(3) + f(4) = 120$, *find* a.

(3) Given that $f(x) = ax^2 + bx$, $f(4) = 8$ *and* $f(1) = -7$, *find* a *and* b.

The teachers were told that the items had been given to top stream ninth grade classes, immediately after they had learned about linear functions and before quadratic functions. For each item the teachers were asked to estimate the percentage of students' success.

They were also asked to justify their expectation by estimation of possible difficulties in the following four categories: unable to get started, use of irrelevant procedure, mistakes in the use of function notation and technical algebraic mistakes.

Student performance versus teacher expectation

Actual student success percentages on the three items and teacher expectation are given below

	Item 1	Item 2	Item 3
Students n = 225	64%	43%	38%
high school teachers	60%	47%	49%
junior high teachers	84%	62%	60%
university student-teachers	16%	13%	10%
college student-teachers	25%	33%	17%

In item 1 for example, the actual and expected difficulties were as follows: ((0) means overestimation and (U) means underestimation)

	unable to get started	irrelevant procedure	mistakes in the use of function notation	Technical algebraic mistakes	
Students	4%	16%	12%	4%	36%
High school teachers	9%	(U) 4%	(0) 22%	5%	40%
Junior high teachers	8%	(U) 3%	(U) 2%	3%	16%
University student-teacher	(0) 35%	(U) 6%	(0) 33%	10%	84%
College student-teachers	8%	(U) 5%	(0) 36%	(0) 26%	75%

The high school teachers are more or less realistic in their expectations. Junior high teachers underestimated student achievement in using the knowledge which they themselves had taught. On the other hand, college student-teachers *overestimated* and university student-teachers completely *underestimated* student ability to even start working on such problems ("it is above and beyond junior high school mathematics"). A discussion of actual student difficulties compared with the teachers' view of these difficulties can provide explanation for these discrepancies in teacher expectation.

Discussion

Note: **Because of the limitation on space, examples given here relate to items 1 and 3 only.**

1. Unable to get started

When students are faced with a "new" problem", for which they do not have an algorithmic solution at their disposal, they can try first to understand the structure of the problem and then decide which of the known techniques to apply. However, even without full comprehension of the problem they can, and usually do, use a well-known procedure relevant or irrelevant, as the case may be, and by doing so may come closer both to the solution and to understanding the problem.

In the three items above the student needs to find parameters of quadratic functions. Substitution of the given data in the functions yields equations which, when solved, give the parameters. The concept of parameter is difficult because of its ambiguous nature as neither a constant nor a variable (Wagner, 1981). On the other hand, the trained mathematician may tend to see in the problem all the implications of the use of parameters and hence conclude that the problem demands a cognitive level beyond the students' ability. Students just recently trained to substitute in functions (in this case, linear functions), can start working and complete the solution, even if they cannot at first (or even at the end) analyse the problem. Although item 3 is more complex to analyse (and to actually solve) than item 1, we see from the results that about the same (high) percentage of students did attempt to solve both of them. About half of the 64% that achieved the correct answer on item 1, did not write a plan, which is some indication of their comprehension of the problem, yet it does not prove that they did not understand. A few wrote comments like, "At first I did not know what to do at all, and it really seemed to be difficult, but after substituting *f(2)*, I could find *a* from an equation and could proceed to f(6) easily." The other half wrote a clear plan; most of them explained and solved the third question as well.

And what did teachers think about students attempting the problem?

The university student teachers analysed the problems from their highly mathematical point of view and thought that all three of them were too complex for the students. At the same time, college student teachers connected the problems with the immediate substitution in functions and did not expect students to be unable to start.

These two groups lack experience and their different opinion probably reflects their educational background. On the other hand, junior teachers *with* teaching experience, expected that fewer students would attempt the third item as compared with the first. The reason they gave is that students would not see the connection with the solution of two linear equations, and thus would avoid the problem completely. The wider experience of high school teachers probably helped them to see that this would not be a serious obstacle.

2 Irrelevant_procedures

All four groups of teachers and student teachers underestimated the impact of linearity, in the immediate past experience of the students, which caused the use of irrelevant procedures (Carry et al, 1980).

Examples for item 1:

- *A proportional solution:* $f(6) = 3 \cdot f(2) = 3 \cdot 29 = 87$

Some of the students who gave this solution were not "comfortable" and in their plan described it as a "simple" solution.

- *A linear solution: finding a linear function through the two points (0, -3), (2, 29).*

One student was very "devoted" to linear functions and even after finding $a = 8$, went on

$$f(x) = 8x + b$$
$$29 = 8 \cdot 2 + b$$
$$\vdots$$

For item 3 about 18% of the students simply found the linear function through the two given points.

3. Mistakes_in_the_use_of_function_notation

Examples for item 1:

- *Wrong substitution:* $2 = 29^2a - 3$; $29 = 2a^2 - 3$

While the first mistake is an expected confusion of *x* and *f(x)*, the second one indicates a complete non-understanding of the meaning of the notation (Dreyfus & Eisenberg, 1982). The latter mistake occurred frequently in a pilot study with lower ability students and needs to be investigated further.

- *Partial substitutions:* $29 = ax^2 - 3$; $f(2) = a \cdot 2^2 - 3$; $f(6) = 36a - 3$

Partial substitution can suggest that the student does not completely understand the process. In fact, in several cases, partial substitution yields the wrong solution. For example

$$\begin{aligned} &\textit{starting with} \quad f(2) = a \cdot 2^2 - 3 \\ &\qquad\qquad\qquad\qquad 4a - 3 \\ &\textit{and then,} \qquad 4a - 3 = 0 \\ &\qquad\qquad\qquad\qquad a = \frac{3}{4} \end{aligned}$$

The influence of teacher education and experience is clear in this issue. High school teachers slightly overestimated these difficulties which they consider as 'serious mathematics, while junior high teachers underestimated the difficulties. (This replicates findings in Zehavi and Bruckheimer, 1983).

Both groups of student teachers strongly overestimated difficulties related to function notation, but from different points of view. The more mathematically trained had doubts about student understanding of the substitution process itself, and expected that students would substitute *2* for *a*, a mistake not found for this top stream population. The college student teachers expected more confusion between x and f(x) than occurred, and also were not sure if students would see the connection between f(6) and the function. Maybe that they themselves feel uneasy with the mathematics and also lack the class experience with such items.

4. Technical algebraic mistakes

Some mistakes in "order of operation" where found in student papers. In item 1, for example: $29 = 4a^2 - 3$

College students teachers predicted that such mistakes would be very popular.

A technical algebraic difficulty occurred in item 3. Students (66%) who substituted correctly obtained a system of two linear equations

$$\begin{aligned} 16a + 4b &= 8 \\ a + b &= -7 \end{aligned}$$

Only 38% achieved the correct answer. About half of the rest stopped at the equations not knowing how to proceed and the other half made ineffectual attempts. The most popular was a naive linear combination $15a + 3b = 15$.

Here again, junior high teachers slightly underestimated this obstacle and college student teachers highly overestimated it.

Conclusion

The significance of this study is that it can help teachers not only to be aware of student cognitive difficulties, but also to be aware of issues where their own conception of those difficulties does not correspond to their reality. In this case study we tried to go even further toward an awareness of some specific reasons for certain misconceptions.

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THE MATHEMATICS TEACHING PROJECT (T.M.T.P.)

CELIA HOYLES POLYTECHNIC OF NORTH LONDON

This paper outlines the background, aims and research methodology of the Mathematics Teaching Project, a three year investigation based at the Polytechnic of North London. Work started on the Project in September 1982 and the results of the preliminary study will be presented at the conference.

The Mathematics Teaching Project set out to provide insight into the range of configurations of teacher characteristics which seem to go together to create good practice in secondary school mathematics classrooms. Good practice was originally conceived as that which stimulates mathematical involvement in pupils (that is, a focus on 'task' and not on 'self' Hoyles 1982, Bishop 1981) and which also provokes positive affective response from pupils.

The Project involves the detailed study of 'good' mathematics teachers from three perspectives:- their own, their pupils and from observation and analysis of their classroom practice.

GENERAL BACKGROUND

The incentive to try to characterise mathematics teachers arose from the finding (Hoyles 1980) that, in the view of pupils, the teacher in the mathematics classroom had a considerable influence on their feelings about the subject and in particular on their confidence and autonomy. In addition, it seemed that such characterisations would be invaluable in teacher education as vehicles for the facilitation of discourse about practice and the incentive for development and change.

There is not at present any comprehensive theory of teaching mathematics so an eclectic approach is adopted in this research which draws on appropriate ideas from psychology, sociology and interaction studies. It is however a belief of the Project that pupil expectations, teacher judgements and classroom interactions are affected by the concern with mathematics, as opposed to another subject area. The Project aims to provide an awareness of what it means to be a good mathematics teacher which is authentic and real. An enhanced understanding of a range of 'particularities' was felt to be the most appropriate means of achieving such an awareness. In addition two further benchmarks have guided the choice of research methodology:- the importance of the individual's own perspective; and the need to bridge the 'gap' between theory and practice, that is, firstly, to identify teacher perceptions that do appear to affect their practice and secondly, to investigate in more detail how these teacher perceptions together with those of the pupil are mediated in and illuminated by actual classroom episodes.

THE THREE PERSPECTIVES

(a) The Teacher Perspective

Despite the controversy over the extent of the influence of teacher expectation on pupil learning (see Hoyles 1973 for review), it would appear, from naturalistic studies, that a teacher's real expectations of their pupils do affect their classroom behaviour. These expectations have also been found to affect the personal qualities and source of motivation a teacher attributes to a pupil (Johnson et al 1963) and that pupils can be affected by such attributions (Brookover 1965, Meyer 1979).

Attribution Theory suggests three dimensions on which the perceived causality of success or failure on a particular task may be classified; that is locus, stability and controllability (Weiner 1979, 1980). Lorenz (1980) suggests that the distribution of teacher explanations of the

6

achievement results of their pupils according to these dimensions can be seen to affect classroom practice.

In the exploratory work undertaken prior to the Project (Hoyles and Bishop 1982), the Rheinberg test was used to investigate differences in reference norm between teachers. The basis of this test (Rheinberg 1977) is that teachers assess pupils achievements as 'good' or 'bad' according to either the average level of the class (social reference norm NO) or to the pupils prior performance (individual reference norm TO) or to a combination of the two. After empirical investigation, Lorenz (1982) reported that differences in reference norm were related to different teacher strategies in the classroom, in particular in the distribution of actions concerned with 'helping' the pupils. Hoyles and Bishop (1982) also found that data from this test (modified for use with English mathematics teacher) produced quite striking differences between teachers which were deemed worthy of further investigation.

A teacher's classroom practice will also be affected by the general frame of reference used as a basis from which pupils are perceived and classified. The study of a teacher's constructs of his or her pupils is a means of probing this frame of reference as it is applied in the context of a mathematics classroom. In the Mathematics Teaching Project a triadic elicitation technique, by means of an interactive computer programme PEGASUS (Shaw 1980), is used. This programme allows flexibility and has the advantage of providing ongoing feedback and analysis, although it is the view of the Project team that for effective use the elicitation by PEGASUS must be accompanied by in depth interview for discussion and reflection.

(b) The Pupil Perspective

Many research studies which have focussed on the teacher have tended to neglect the views and perceptions of pupils and their interpretation of classroom events. The way a pupil responds to the teacher will in part be affected by his/her expectations of and judgements of that teacher. In line with the overall stance of the Project, it is intended not only to elicit the pupil views of what a mathematics teacher should be like and to find out how their teacher compares with this ideal, but also to discover the ways these teacher characteristics are manifested in the pupil's view) in the classroom; that is, actual episodes, recalled by the pupil to exemplify a certain 'good' teacher characteristic, will be collected and examined. It is through this means that it is intended that pupil data will illuminate and be illuminated by observational data.

(c) The Mathematics Classroom

The problems associated with the analysis of classroom practice has been widely documented. Classroom talk is by its very nature multiple and indefinite. It is however fundamental to the Project that any attempt to characterise mathematics teachers tries to get to grips with the 'we-relation' between teachers and pupils in the classroom (Hargreaves 1977 P282). The theoretical stance of the Project is to develop and make public the parameters within which observations are made and the rationale for any choice of illustrative extracts.

PRELIMINARY STUDY

With this background in mind a preliminary study of two teachers was carried out during 1982-83. Classes singled out for particular attention were of 13 - 14 year old pupils top and middle/low sets. The following investigations were undertaken:-

The Teacher Perspective

- Elicitation of the personal constructs of their mathematics pupils
- Cluster analysis of these constructs
- Reflection and discussion on the nature of the constructs and clusters
- Elicitation of the teachers' attributions of success or failure in a mathematical task for each individual pupil
- Analysis of these attributions in terms of the three dimensions of perceived causality
- Test of the reference norm and calculation of the teacher typification on the ideographic/normative continuum.

The Pupil Perspective

- Elicitation from all the pupils in the experimental classes of written descriptions of an 'ideal' mathematics teacher
- Extraction from these descriptions of up to ten of the most frequently mentioned factors
- Ranking of the above factors by the use of a paired-comparison test completed by all the pupils
- Grading of the teachers by each of their pupils on a numeric scale for each of the factors
- Individual interview of each pupil in order to obtain a description of classroom events chosen by the pupil as practical manifestations of one or more of the positive characteristics which they had attributed to their teacher.

The Mathematics Classroom

- Weekly observation and audio-taping of the two teachers over a period of two terms
-'Reconstruction' of a selection of lessons in which observational notes of teacher comment (prior and post lesson), actions or non-verbal gestures are co-ordinated with sections of transcript in order to 'bring the lesson to life'.

- Analysis of the reconstructions in terms of the following ten categories:-

Mode of teaching: - Instructional or interactional; Type of questions; Influences on content and pacing; Assumptions made by the teacher; Repetitions; Vocabulary of the lesson, Pupils' language; Teacher's use of praise; Distribution of teacher's interaction between pupils; Identification and analysis of episodes of mathematical communication and development.

- Comparison of the two teachers in terms of the above categories.

The first stage of the main study is being undertaken in the summer term 1983, in three mixed London comprehensive schools well known for their enthusiastic and competent Mathematics Departments. The teacher sample for the main study consists of nine teachers highly regarded within their schools and by the staff of the Polytechnic of North London. Two fourth year classes are to be studied for each teacher, one of high ability and one of middle/low ability. A further similar round of data collection will be undertaken in the summer of 1984 with the same teachers and types of classes.

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MATHEMATICS INSERVICE THAT WORKS:
A RESEARCH-BASED MODEL

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In the current U.S. educational milieu mathematics and science inservice education is receiving greater attention than ever. Although staff development programs are expensive in terms of money and time, the need for them has increased because of the loss of qualified teachers to industry and the stabilization of teaching staffs. The decrease in the number of new teachers entering the system with up-to-date knowledge of subject matter and/or techniques (McLaughlin & Berman, 1977; Buder, 1977) adds to the problem. The cost of inservice programs may be justified to a critical public only if they result in demonstrated student improvement.

Researchers have analyzed staff development programs from all parts of the country with great care. Those programs which were considered successful were found to possess similar characteristics. This paper will detail these characteristics and then describe a nationally-validated mathematics inservice program based upon them.

EFFECTS OF INSERVICE ON STUDENT ACHIEVEMENT: Studies of the effects of inservice education on student achievement show that a positive relationship does exist. Statistically significant student improvement has been found to occur as a result of teacher inservice, irrespective of grade level or subject area (Berman, 1981; Fitzmaurice, 1976; Good & Grouws, 1979).

One frequently-used method of conducting staff development is the multiplier or turnkey approach. This strategy is considered highly appropriate for use in areas where many teachers from different buildings (or districts) require the same inservice training. It involves the training of selected teachers in a particular topic and their subsequent employment as inservice workshop leaders for others in their school or district. The multiplier strategy has the potential of providing for staff development, utilizing local resources, to train a large number of personnel within a relatively short period of time (Tobin & Dye, 1977; Wirtz, 1974).

Researchers evaluating student achievement as a result of teacher inservice based on a turnkey approach found that statistically significant growth in knowledge occurred in students of both the original trainers and their trainees (Berman, 1981; Dilworth & Warren, 1973; Lawson, 1978). The multiplier strategy thus appears to be an efficient, cost-effective staff development technique. It provides an opportunity to key inservice training to the needs of the individual school building and may be directed by a teacher from that building who understands the unique problems faced by his/her colleagues.

EFFECTIVE INSERVICE THROUGH EXPERIENTIAL LEARNING

Cognitive psychologists assert that learning takes place through the interaction of the individual with materials, peers and the instructor. Learning is described as a developmental process, with each new concept built upon previously-existing knowledge. Individuals learn by manipulating objects in their environment, then by reacting to representations of these objects and, finally, operating with the abstractions that result from these experiences.

Empirical data on adult cognitive development is still rather sparse. However, some researchers have documented evidence which supports the need for experiential learning at the adult level (Bender & Milakofsky, 1982; Joyce & Showers, 1980; Oja, 1980; Wood & Neill, 1978). Apparently, a large proportion of adults are operating at the concrete operational, not the formal, stage of intellectual development. A discovery-oriented, problem-solving approach to learning thus appears to be as important for adults as it is for children. This assumption is supported by Richard Skemp in his book on the psychology of learning mathematics.

> We all have to go, perhaps more rapidly than the growing child, through similar stages in each new topic which we encounter . . . . The mode of thinking available is partly a function of the degree to which the concepts have been developed in the primary system. One can hardly be expected to reflect on concepts which have not yet been formed, however well-developed one's reflective system (Skemp, 1971, p. 66).

What the foregoing suggests is that teachers should learn new concepts and skills in an active learning environment using concrete materials when needed. Inasmuch as teachers tend to teach the way they were taught (Dilworth & Warren, 1973; Fuson, 1975) as well as what they were taught (Goodlad, 1983) the experiential approach to staff development assumes great significance. Appropriately-designed inservice education can become the model for increased use of hands-on activity-oriented programs which introduce new subject matter to elementary students.

CHARACTERISTICS OF EFFECTIVE INSERVICE: As a rule, effective inservice programs involve a continuous theme, over a period of time, interspersed with classroom tryouts. The successful programs have immediate applicability in the classroom; they combine subject-matter content with teaching methodology. Activities for teachers that parallel those to be used with children are employed, thus creating a model for subsequent classroom application (Berman, 1981; Friederwitzer, 1981; Nicholson, et al, 1976).

The goals of inservice education must be carefully considered before the program begins. Teachers should be provided with the time, the opportunity, the means and the materials for improving professional competence. They should receive assistance with the development of creative instructional approaches that are meaningful and appropriate to their students. Training in the implementation of innovative curricula or instructional practices and help in applying new insights into the learning process to themselves and their pupils must also be available (McCormick, 1979; Tye & Benham, 1978).

PROJECT SITE, AN EXEMPLARY STAFF DEVELOPMENT PROGRAM. The National Diffusion Network is a nationwide system established to assist schools, postsecondary institutions, and others to improve their educational programs through the adoption of already developed, rigorously evaluated, exemplary education programs. The Network consists of over one hundred programs for all grade levels and serving all disciplines. Project SITE: (Successful Inservice through Turnkey Education) was nationally validated by the U.S. Department of Education and has been part of the National Diffusion Network since 1982.

The purpose of Project SITE is to provide inservice training in mathematics content and appropriate methodology to elementary teachers (grades 2-6). Since elementary teachers are often poorly prepared and/or uncomfortable in mathematics, they frequently "resist" inservice that is designed to teach them mathematical content. They may resent the implication that they "don't know" the subject and need more training, even though it is true. Research shows that measurement is one of the most poorly taught areas of math, usually left for the end of the school year. Therefore, Project SITE uses measurement as the vehicle of instruction for teaching many mathematical concepts.

Participants interact with manipulative materials, taking an active role in their own concept development and learning of mathematical skills. A unified, cohesive program is created out of normally disparate components through the integration of mathematical content and teaching strategies based on learning theory with practical classroom applications. This type of course has not traditionally been part of the preservice or inservice training of a majority of elementary school teachers. Thus, the dissemination and implementation of the SITE program becomes the means for providing instruction in mathematics content and appropriate methodology to elementary teachers in an innovative, nonthreatening way.

Background: Project SITE is an outgrowth of a program originally developed at the Rutgers University Graduate School of Education, New Brunswick, NJ from 1976-80. The Rutgers program responded to a U.S. Office of Education initiative for metric education programs. As metrics was unfamiliar to most United States elementary school teachers, an in-depth staff development program was needed before implementation of metric activities could begin with pupils. The multiplier (or turnkey) strategy was the basis for an inservice interventional plan which operated successfully in urban and suburban school districts spanning a variety of socioeconomic levels. Since 1980, the program has been disseminated nationally by Educational Support Systems, Inc.

Design: The SITE program was designed to incorporate the characteristics of effective inservice programs described above. The inservice guide, Measurement in the Elementary School, was written specifically for this program. It was based on developmental theories of learning espoused by cognitive psychologists. The theoretical framwork of the program also includes recent findings on teaching/learning styles, experiential learning for adults and research by mathematics educators on appropriate models of instruction for teaching new mathematical ideas.

METHODOLOGY AND CONTENT: The methodology focuses on problem-solving through guided discovery with manipulative materials. Each new section of mathematics and measurement builds upon concepts developed in preceding sections to evolve an integrated, systematic understanding of the mathematics included.

Effective measurement instruction incorporates a number of mathematics concepts and skills i.e. estimation, place value, decimals, and geometry. The Project SITE curriculum includes these as well as a number of other topics such as graphing, statistics, ratio and proportion, which are frequently omitted from elementary mathematics instruction. Emphasis is placed on in-depth development of area, perimeter and volume, sections of the geometry curriculum frequently misunderstood by many elementary school teachers. Activities are sequenced to develop basic concepts of covering and filling through the discovery and application of formulas.

Since teachers tend to teach the way they were taught, as well as the way they learn, the program incorporates activities for teachers which parallel those suggested for children. The teaching methods used by the project trainers, the manipulative materials employed at the workshops and the printed matter distributed, are all intended to serve as models for subsequent use by participants.

The SITE program: Participants receive four full days of mathematics instruction. Each district or school is requested to send at least one supervisor to attend the training series alongside their teachers. The participation of principals and/or district supervisors is considered crucial. Their presence and involvement in the inservice sessions is an indication to their teachers of the importance placed on the training. Furthermore, their attendance enables them to develop a working understanding of the content and methodology of the program so that assistance and support can be provided during the implementation phase of the program. As the Rand report aptly states, principals are "the gatekeepers of change" (Berman & McLaughlin, 1978), who can enhance or destroy a new project.

Upon completion of the initial training sessions, the second phase of the inservice program begins. This phase involves training the other teachers in each participating school building or district via the multiplier approach. The original participants, or turnkey teachers, conduct inservice programs for their colleagues using a detailed "Script for Workshop Leaders" which parallels the activities used by the original workshops. This strategy allows the program to multiply the benefits of the original inservice experience, without a concomitant increase in cost.

The presence of turnkey trainers in each participating building permits the program to respond to individual school needs. The turnkey teacher is able to tailor inservice to the particular needs of his/her colleagues. All teachers, turnkeys and colleagues, are expected to implement the program with students using the easily-replicable activities through which they themselves learned.

EVALUATION: All participating teachers and students are pre- posttested. The criterion-referenced tests employed were developed for the project and have been found valid and reliable. The efficacy of the SITE project has been measured by the growth in knowledge (i.e. increased achievement) of teachers and students. During each year of operation, all participants showed statistically significant gains from pre- to posttest at a minimum of the 5% level. This staff development program has thus proven to be an efficient means of providing mathematics instruction to thousands of teachers and, through them, to tens-of-thousands of children.

IMPLICATIONS FOR STAFF DEVELOPMENT: The rationale underlying the development of the instructional materials and procedures used by Project SITE was derived from theories advocated by cognitive psychologists as to how learning occurs. The successful implementation of the program over a period of several years suggests that instructional formats for adult learners which follow developmental, sequential patterns, incorporating activity-oriented approaches, can increase subject matter knowledge while simultaneously modelling future behavior. Through their participation in the SITE inservice workshops, the turnkey teachers learn mathematics and measurement content as well as appropriate teaching strategies. The new knowledge and instructional techniques are utilized subsequently for training colleagues and teaching students.

Aside from the positive results described, this approach to staff development offers residual benefits to participating school districts. Through the training of personnel already on the job, a cadre of staff developers available for future service is created. The presence of residual resource personnel in each school building also permits "individualization" of the program. The resource personnel can tailor the contents and methodology of inservice sessions to the particular needs of the school. Additionally, they are available to provide immediate assistance and advise colleagues as problems arise.

Utilization of the multiplier strategy thus becomes a cost-effective tool enabling school districts to provide expanded staff development programs without a concomitant increase in funds. The initial training costs incurred in developing a cadre of resource personnel are more than repaid in future years as the trainers are called upon by their districts to condict additional staff development programs. Furthermore, the benefits of this residual inservice cadre will be multiplied year after year, as the trained teachers continue to implement classroom programs with their students.

The project format, with its multiplier strategy, is adaptable to a variety of content areas, further enhancing the utility of this model as an instructional vehicle for teachers and, subsequently, for students. Adaptation of this inservice model would permit school districts to implement cost-effective staff development programs with the reasonable expectation that student achievement would increase as teachers gained knowledge of new subject areas and appropriate teaching strategies.

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DIDACTICAL INFORMATION SERVICE FOR MATHEMATICS IN SCHOOL (5th grade – 10th grade)

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WHY A DIDACTICAL INFORMATION SERVICE?

In the mathematical classroom teachers and pupils have a lot of problems with math and with one another. W.Breidenbach, an old experienced teacher trainer and book writer, says: " we have tried to do nearly all things for better teaching of math, but it didn't help. Therefore the causes for the 'math gap' in the classroom must be deeper than we had thought up to now."

An increasing minority of teachers has the same feeling as Breidenbach, but they can't find any help in their methodic books and in the frames given by the administration. Therefore a new sort of information about the teaching-learning-processes seems to be necessary.

In 1979 a small team of the Landesinstitut Nordrhein-Westfalen Neuss (FRG) started the DIDACTICAL INFORMATION SERVICE FOR MATHEMATICS IN SCHOOL (DID-M). It is a series of textbooks and documentation books referring to the most important topics of math for about 5th grade to 10th grade. The project will be ended in 1986; afterwards a supplementary service is planned.

POSITION OF DID-M

Most of 'math makers' in school and for school do this on the background of their own experience, and their scientific knowledge. During the teacher training periods the subjects being discussed often only are " how can teaching be made more effective, more interesting etc?"

But in classroom reality teachers often are getting disappointed because the pupils' problems have nothing at all to do with what the teachers had learned. Teachers then say: "pupils are not interested, they are against me, school politics are too liberal etc."

DID-M tries to objectivate the situation for teachers.

Firstly it informs about the communicational view of teaching (Bauersfeld 1983); on one side there are the teachers' concepts and on the other side the concepts of the learners. Every so often there are gaps between these concepts; teachers and learners "speak about the same but do not mean the same" (Krummheuer 1983).

Secondly DID-M informs about the attitudes and contents of teachers' concepts and learners' concepts as known up to now.

Thirdly DID-M gives hints for the teachers how to bridge the gaps and for administrators how to renew the curriculum for better bridging the gaps.

DESIGN OF DID-M

Teaching concepts (related to the FRG schools) are described in the textbooks in view of historical tradition, scientific influences, administrational frames, and didactical transformations. By questioning teachers and searching for results of classroom research the actual teaching situation is described (Andelfinger,1981).

You will get some information about learning concepts by questioning teachers. Most information results from material of educational research. DID-M makes systematical searches for such material in national and international information pools (ERIC, Psychinfo, FIZ,..) and by contacts with researchers. Comparing these results gives us an imagination of important learning concepts.

In a third step the teaching-learning situation can be shown by comparing the teaching and the learning concepts and by analyzing connections between them. This step includes also hints for teaching (in a somewhat hypothetical form).

All this information is given in the textbooks of DID-M.

The background materials (questioning) and references (i.e. abstracts) are given in the documentation books, as well as the results of evaluating the DID-M-books (see later).

PILOT STUDY "PROPORTION"

Design and products of DID-M have been tested in a pilot study for the topic "ratio/proportion" and its surroundings in school mathematics (see references). Some results:

The teaching concepts show a great difference between theory and practice. Ratio and proportion - didacticians say - have to be a headline and an important topic. Practice in schools is very different from this. Here ratio and proportion are nearly isolated in geometry (similarity); fractions are operators or regional sets; rule of three is a part of linear function theory or of equation theory; probability and statistics have not yet got a real place in the classroom.

About 5% of the pupils overtake these teaching concepts.

A majority of pupils has other concepts:

- "rule of three"-problems are solved by isomorphic strategies, additive or multiplicative, avoiding fractions. Functional and/or proportional strategies are very seldom and unstable.
- pupils make a difference between 'basic' fractions (e.g. 1/2, 1/4) and 'artificial fractions' (e.g.7/13). For pupils basic fractions are instinctively numbers, artificial fractions are puzzles with vertically written pairs of numbers or tasks without results. Formal cognitive concepts play an outstanding role in computing with fractions.
- pupils have an unreflected imagination of 'ratios'. There are hierarchical stages to come from 'ratio' to 'proportion'. How pupils climb these stages is not yet known enough. Only few pupils reach the higher stages. Concrete models and experiments can help to climb the stages; information processing strategies are also helpful.
- similarity has three main aspects: parallels, angles, proportion. The proportional aspects of a pair of figures are not the most important thing to recognize similarity.

The pilot study shows that proportional concepts do not grow without impulses nor do grow in the field of artificial fractions. Proportional concepts often correlate with the age of the pupils and their intelligence. Proportionality as a linear function is a very complex concept; anti-proportionality has nearly no connection with proportionality.

EVALUATION OF THE PILOT STUDY

The textbook "proportion" has been evaluated by a scaled questionnaire. The testgroup agreed with the content of the book and mentioned the high correlation with their own (often secret) experience in the classroom. Most of the persons said that the book will influence their teaching and their role as teachers.

The reactions of administrations were very different. Some used the book for their planning, others blocked the spreading of the DID-M-books in their district.

Educational researchers were very happy with the documentation books of DID-M.

The DID-M project initiated a controversary discussion in groups of teacher trainers.

THE STUDY " ARITHMETICS/ALGEBRA/FUNCTIONS"

In 1982 DID-M started the second step of its work, related to the non-geometrical topics of school mathematics from 5th to 10th grade.

The analysis of teaching concepts again shows a great difference between theory and practice. The administrational and didactical frames are characterized by the long time line 'constructing number areas from IN over IQ to IR' and by 'constructing a logical building of open statement algebra'. Teachers' practice in school is characterized by fighting against difficulties with numbers and with manipulative algebra. What a stupid situation!

So far DID-M does not give any ready learning concepts but suggests some working hypothesis:

- There seems to exist a big pool of fundamental attitudes, strategies, and concepts, containing formal/cognitive elements, information processing elements and kinds of understanding variables and numbers.
- the area of numbers seems to be splitted in an area of numbers written in ciphers, basic fractions, and artificial fractions.
- algebra seems to be splitted in a hierarchy of three alge-

bras, the 'number-operation-algebra', the 'try-and-put-in-algebra', and the 'formal-syntactic-manipulative-algebra'.
- the theory of functions seems also to be splitted , some parts being combined with the upper mentioned algebras.
- sharp boundaries seem to exist between some parts of arithmetics and some parts of the algebras and no sharp boundaries between other parts of these two.

The results of the second step of DID-M will be published in 1984. The last step of DID-M will be done in the field of geometry.

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DOES THE GROUPING OF STUDENTS MAKE A DIFFERENCE -

ON THE PSYCHOLOGY OF TEACHER-STUDENT INTERACTIONS IN MATHEMATICS EDUCATION

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Usually, the basic assumption is that changes in the educational system cannot be brought about without the active cooperation of the teachers. 10 years ago, a reform of Sekundarstufe II (ages 16 to 19) was decided upon in the Federal Republic of Germany by an administrative body which intends to establish rules for all matters of education in the country. Since that time, students belonging to the age group 16 to 19 have had the choice between a basic and a talent course in mathematics. These courses differed, in the first place, simply by the number of lessons per week the subject was taught. Thus, students belonging to the basic course received three, students of the talent course five or six lessons per week.

There is no clearly defined educational conception for the various types of courses. The question, therefore, is how the teachers react to, and cope with this situation. A project conceived three years ago set itself the following three main tasks in investigating these questions :

- analyzing the content of textbooks, regulations and syllabi with the objective of finding out whether these documents, which are pertinent for mathematics education, offer different paths for teaching mathematics in basic and talent courses;
- doing representative surveys of teaching styles and attitudes of the teachers affected by the reform, with the objective of establishing interactions between teaching styles and attitudes on the one hand, and course type on the other.
- carrying out case studies in selected schools with the objective of establishing the interactions between the individuals involved in mathematics education, i. e. case studies oriented towards an ecological perspective of teaching research.

My contribution to this conference shall merely be concerned with the second of these tasks, as presenting all three would exceed the time limit set. It must be emphasized, however, that all three parts of the project are interconnected: it is only the content analyses of the documents pertinent for mathematics education which permit us to delimit the frame conditions within which the teachers' teaching styles and attitudes develop. And only classroom studies make teachers' and pupils' concrete behaviour the object of observation.

Hence, our representative survey's objects were teaching styles and attitudes, in particular among mathematics teachers of Skundarstufe II. Our interest, in that context, was in the social organization of learning, so we took up an approach which was originally developed by British educational sociology (BERNSTEIN, 1977) and has since been transformed for the purposes of mathematics education by PFEIFFER (1981). According to that approach, both the cognitive and the social organization of knowledge are subject to principles which are acquired by the individuals concerned, and according to which the latter behave in the classroom.

BERNSTEIN characterizes these principles by a pair of idealized opposites, the collection code and the integrated code, which can be briefly described as follows:

The collection code implies strict hierarchies, boundaries and little control exercised by all individuals concerned over type, structure, temporal order of knowledge and teaching organization.

The integrated code implies abolition of hierarchies, crossing boundaries, and a high degree of control exercised by teachers and learners over type, structure, and temporal order of knowledge and teaching organization.

The items of these two codes assist us in listing the characteristics of knowledge taught in an organized, specific, and more concrete form. The best structure of these characteristics was obtained by introducing the following dimensions (beginning with those referring mainly to the knowledge side of teaching and terminating with those whose emphasis is on the organization of teaching) :

(1) Type and structure of the knowledge to be taught (the collection code's hierarchical organization and strict boundaries versus the integrated code's tentative organization of knowledge and integration of knowledge and subjects);

(2) Relationships of teachers and learners towards the knowledge acquired and to be acquired (strong involvement with the subject and a conception of knowledge as an object of teaching in the case of the collection code versus less pronounced involvement with the subject and a conception of knowledge as of something which can be used and applied in the case of the integrated code);

(3) Organization of teaching with regard to subject matter (schoolmasterly teaching and relatively little control exercised by teachers and learners over selection, organization, pace, and temporal order of subject matter in the case of the collection code versus an emphasis on self-organized acquisition of knowledge and a high degree of control of teachers and learners over selection, organization, pace and temporal order of subject matter in the case of the integrated code);

(4) Organization of teaching with regard to the pedagogic principles applied in the classroom (a hierarchical order of relations of authority and very ritualized relations between teachers and learners in the case of the collection code versus an egalitary structure of the relations of authority and the intention to enable teachers and learners to develop new social relationships in the case of the integrated code);

(5) Institutional and material frame conditions (strongly structured concept of classroom arrangements, together with exact boundaries and delimitations of the subject and detailed instruction from outside (rules, regulations) in the case of the collection code versus a fixed, variable conception of classroom arrangements and few instruction from outside in the case of the integrated code) .

Discussions within the Bielefeld working group "Mathematics Education in the Sekundarstufe II" at the Institute for the Didactics of Mathematics were concerned with specifying this theory of codes with regard to mathematics education. The following representation is a brief summary of this attempt:

Three questions can be considered important for distinguishing between various types of mathematics teaching :

- Is the emphasis on training mathematical methods and exercising calculating skills, or on developing mathematical problems?

- Is teaching alligned to traditional school mathematics (i. e. mostly intramathematical), or is there an attempt to develop a comprehensive orientation in students which stresses both crosslinks between the various fields of mathematics and mathematics' richness in relationships in application situations?

- Is the emphasis of the teacher's activities on the process of learning, or on the product of learning?

If we attempt to locate BERNSTEINs codes of the knowledge taught in school, the collection code is characterized by an orientation toward the product of learning and a restricted alignment to school mathematics. As opposed to that, the integrated code in school mathematics can be described as a comprehensive orientation towards a mathematics rich in relationships, towards developmment of mathematical knowledge, and towards the process of learning.

If we now resume the initially described differentiation according to basic and talent courses, a first hypothesis could be stated :

The basic course practises a type of teaching which can be characterized by the attributes of the collection code. While the talent course is not determined by the attributes of the integrated code, it shows statistically significant deviations towards the integrated code in all of the three dimensions described.

In the frame of the representative survey of teaching styles and attitudes pertinent for mathematics education, we developed a questionnaire which was adapted as closely as possible to the mathematics teacher's classroom activities. This was meant to reduce the danger of having the teachers merely respond in an manner they imagined we thought they should respond. In this matter, HOPF (1980) gave us some important ideas.

The questionnaire was revised several times, a pretest was carried out in 10 federal states, and the final version was posted to all schools within the Federal Republic having Sekundarstufe II (a total of 2419 schools) at the beginning of the school year 1981/1982. An extensive system of reminders helped us to get back the questionnaires from 2063 schools. This is a response rate of 85.3 percent. The sample can be considered as representative for the population we had adressed (see REISS, 1983).

The mathematics teachers were asked to answer all items with reference to the course they had held in the second term of 1980/81. Besides some items concerning the person of the respondent and his or her general image of mathematics and mathematics education (presage variables according to DUNKIN & BIDDLE), most of the approximately 300 items refer to classroom activities within the course. These belonged to the following fields :

- establishing subject matter
- exercise tasks and problem tasks in the classroom
- revision
- homework
- organization of teaching and students participation
- assessing performance, and
- frequency of the use of teaching aids.

There are two versions of the questionnaire: one for the basic, and one for the talent course. The sole difference is that the term of "basic course" in the former was substituted by "talent course" in the latter. This specification of the type of course occurs very frequently among the items in order to keep the mathematics teacher from answering the questions in a general way instead of referring to the specific classroom activity in his own course.

As most items were concened with concrete classroom acivities, our interest was how frequently these occurred within the entire halfyear surveyed. For this, we presented a scale containing six categories, with the extremes of "always" and "never".

The results were first broken down according to individual items; after that questions of establishing scales and of factor structures were discussed. To state the most surprising result first: with regard to the aspect of the teaching styles investigated, the fact whether the students are given three or six lessons per week seems to make little difference. Merely with regard to the dimension of "development versus standardidization of mathematical knowledge", there were clear-cut differences between basic and talent courses. Some possible conclusions for mathematics education shall be given in the oral contribution.

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EXPERIENCES CONCERNING MATHEMATICAL EDUCATION IN AFRICA

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The author had the occasion to spend as a visiting professor, in the years 1982 and 1983, some time at an african university (Kisangani, Zaïre) and believes that it is worth while to report about his experiences on mathematics instruction in an african country. Scientific life in Africa is very different from ours in Europe, most problems arise from the fact that material conditions of life and means of communication are much more intricate than in Europe.

The educational system in Zaïre ressembles very much the belgian system. Primary and secondary level instruction both take six years, from 6-12 and 12-18 respectively. After secondary instruction, a restricted number of students is selected for entering university. Due to material conditions, the number of available places at the universities is always inferior to the number of applicants. Hence, a selection is unavoidable. My experiences concern first year students in biology and agronomy on a course of Introductory Mathematics, preparing for statistics and applications of mathematics in biology and agriculture. The content of this course was a very classical one : elementary geometry, linear algebra and calculus. Although the basis on which impressions and conclusions are grounded, is rather small, it is justifiable to say that it is a caracteristic sample out of a general situation.

The learning difficulties the students have to face can be classified in two major categories : difficulties due to external factors such as social life and pedagogical situations and others due to internal factors, which are mostly related to some unacquaintance with classical features of our european way of thinking and conception of scientific culture. The first category seems to be preponderant, to such an extent that it is sometimes impossible to decide wether a lack of knowledge is due to a failure of understanding mathematical reasoning or to a lack of familiarity with verbal expressions and with real life examples that often go together with mathematics teaching.

1. Conditions deriving from the facts of social life.

Becoming a university student in an african country means to be thrown to one's own resources. In fact, students came from all over the country, some of them had their home about thousand kilometers away from the university. They are in the impossibility to return to their family more than once a year. Lodging and board is supplied by the university; inevitably, all this is very uniform in character and the small financial means of most students do not allow any exceptional expense. A student's day is rather monotonous and material problems, together with the effects of the climate, often prevent from intellectual efforts.

In the same context, it is worth while to note that teaching makes use of a foreign language. Although French is the official language, and most students accomplish the task to master it rather well, it is not their native language and it occurs that they have some difficulties to grasp all subtleties of a complicated paragraph, e.g. commenting a mathematical formula in a verbal form requires an effort to avoid an all too intricate terminology.

2. Pedagogical situation : far from optimum.

The most crucial problem is how to get qualified teachers at the university level. There are not enough native professors available in order to assume all teaching charges. The open places are taken by foreigners, mostly from Europe, some are full-time members of the local staff, others are visiting professors, coming for a short time and teaching a course at high rate. It is not exceptional that a visiting has two hours of lecturing, followed by two hours of exercises a day, finishing his course in this way in four or five weeks.
In the meantime the other courses are reduced or stopped. From

a didactical viewpoint, this is a rather unsatisfactory situation. Long communication lines are responsible for heavy restrictions on the availability of teaching aids, didactical equipment and books. All this has te be imported over long distances; especially the constitution of a suitable library is an unsolved problem. Books are expensive and information about new volumes comes in at slow rate. As a result of this conditions, students often have nothing else but their classroom notes in order to prepare the examination.

3. Internal factors.

It will not be surprising that sudents showed an unfamiliarity and even opposition towards abstraction and formalism. This is a common feature of all courses for non-professional mathematicians and there is no need to go further into this subject here. It suffices to realise that african students are educated in a society which tends almost exclusively towards pragmatism.

Related to the foregoing is a definite preference to verbal expression versus formal and concise statements by means of a formula. Let us recall that students are linguistically well gifted, speaking several languages; the tendency to convert anstract definitions into litterary sentences is probably another aspect of this ability. The real problem is to explain why the content of a verbal statement may differ from the meaning (concept image) incorporated in the formal (mathematical) expression.

In a general setting, the teacher has to take care of refering to situations that are part of the general european background, but that are often unknown in Africa. An example of an unsuccessful attempt to explain the notion and the fundamental laws of probability theory, was the use of card playing. It turned out that several students were unfamiliar with card playing, and that the distribution choosen as a starting point for calculations didn't mean anything to them.

Nevertheless all this problems, some positive facts have to be mentioned. Students are aware of the fact that they are lucky to have an opportunity to participate in higher education, they are very obliging and eager to accomplish their tasks.

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RESEARCH AND CURRICULUM DEVELOPMENT: THE DEVELOPMENT OF DIAGNOSTIC ASSESSMENTS AND TEACHING/SELF-LEARNING MATERIALS IN MATHEMATICS FOR ADULTS.

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INTRODUCTION

This paper outlines the results and procedures of a project concerned with the development of parallel diagnostic assessments in basic mathematics together with complementary teaching/self-learning materials for TOPS trainees. TOPS trainees are adults on Training Opportunity Schemes which are organized by the Manpower Services Commission (MSC), a British Government Agency, which offers more than 500 courses nationwide from bricklaying to electronics for people 19 years of age or older. The vast majority of these courses take place in establishments called Skill Centres.

There is concern in Britain about the mathematical difficulties and needs of adults. Researchers over the past decade have addressed a number of relevant aspects dealing with:

(i) Mathematical difficulties experienced by a variety of groups of people, e.g. Sewell (1981), Reys (1976) have considered adults in general; Rees (1973), Hitch (1978), Barr (1980) have considered apprentices;

(ii) the mathematical requirements of adults, i.e. the work of Freshwater (1981), Howie (1981), Lindsay (1977), Fitzgerald (1976) and

(iii) the methods of teaching numeracy to adults, e.g. ALBSU (1982), BSU (1982), Riley and Riley (1978).

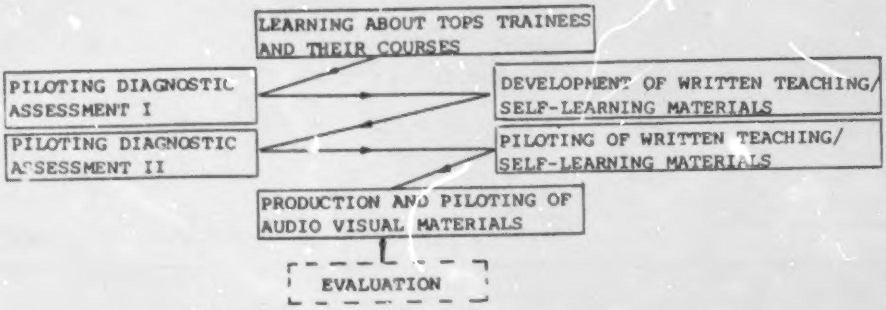
Many of the ideas and findings of the works referenced above were utilized in the project.

Staff within the MSC were not satisfied with the resources available for the selection of TOPS trainees for revision of basic mathematics. They wanted an assessment that would highlight the concepts and skills that needed revision and they wanted teaching materials that could be used for revision.

The teaching/self-learning materials consist of eight written units/sections with a brief instructors' guide, together with seven Audio Visual packages.

PROCEDURE

The procedure is best summarized in a crude flowchart.


LEARNING ABOUT TOPS TRAINEES AND THEIR COURSES
PILOTING DIAGNOSTIC ASSESSMENT I
DEVELOPMENT OF WRITTEN TEACHING/ SELF-LEARNING MATERIALS
PILOTING DIAGNOSTIC ASSESSMENT II
PILOTING OF WRITTEN TEACHING/ SELF-LEARNING MATERIALS
PRODUCTION AND PILOTING OF AUDIO VISUAL MATERIALS
EVALUATION

THE RESULTS OF TESTING

We shall consider these results in two ways: (a) from the trainees' viewpoint (i.e. difficulties experienced) and (b) from the test statistics standpoint (means, reliabilities, etc.).

a) From the Trainees' Viewpoint

The difficulties experienced by trainees in both tests are summarized in the table below.

TOPIC	DIFFICULTY
Place Value:	Knowing the meaning/value of the digits to right of the decimal point.
Division:	If there is a zero embedded in the answer this tends to be omitted.
Squares and Square Roots:	a) Of integers (i) Confusion of the term square and square roots. (ii) Lack of understanding of the magnitude of a square root. b) Of numbers less than one.
Adding of Common Fractions:	Adding tops and adding bottoms is a popular method used.
Conversion of Common Fractions to Decimal Fractions (to 2 decimal places):	(i) Misunderstanding the meaning of "correct to 2 decimal places". (ii) Giving the top number, decimal point, bottom number as the conversion.
Multiplication and Division of Numbers Less than One:	Confusion of methods of addition/subtraction with multiplication/division (i.e. decimal point alignment).
Solution of Equations where the Unknown is the Denominator of a Fraction.	
Substitution:	x^2y appears to read as x2y or $x^2 + y$.
Use of Scales from a Map.	
Ratio	Area α dimension^2; volume α dimension^3.
Conversion of Square Metres to Square Millimetres.	
Finding the Circumference and Area of a Circle.	

b) From the Test Statistics Viewpoint

Briefly, the information collected suggests that the two samples of 169 and 153 trainees respectively, were from similar backgrounds with respect to age of trainees, trades, mathematical background, etc. Further that their performance in the tests was comparable in terms of mean scores and extent of difficulty experienced on specific items.

The statistical reliability of the two tests using KR20 was 0.92 of both tests which is acceptable and consistent with results from previous studies, as are the reliabilities for the component topic areas within the test.

THE RESULTS OF PILOTING THE WRITTEN MATERIALS

The materials were developed in topic sections,each topic forming a unit,so that they could be used as a source book for class instruction or as self-learning materials. There are eight units in all: Place Value; Multiplication and Division of Natural Numbers; Multiplication and Division of Decimal Fractions; the four arithmetic operations on Common Fractions; Ratio; Algebra; Squares and Square Roots; Shape, Perimeters, Area and Volume. The structure of a unit is: example and exercise, example and exercise, ..., end of unit exercise.

Teaching in general appears to involve either class teaching or self-learning using programmed texts. The method of instruction used at the Skill Centre where the materials were piloted was a tutored self-learning approach. It was decided to conform to the Skill Centre practice, so that the materials were piloted as self-learning materials.

It was clear that the researcher would only be dealing with small numbers and be in an unrealistic situation if he only dealt with one intake of trainees. It was decided that two consecutive weeks intake should be used for the pilot study. All trainees entering during this period were given Diagnostic Assessment I. If they scored 50% or below they were candidates for revision. Five trainees were selected from the first intake and four from the second.

The whole set of materials was used for the standard 20 hours revision time on all the trainees over a period of 6 weeks. It was important that the trainees were exposed to all the materials since there were so few of them and information would be lost on the optional units if they were allowed to be optional(!) The Skill Centre time-tabled three sessions of revision a week. They lasted for an hour to an hour and a half.

The materials appeared to be well received. The trainees worked through the units at their own pace. Some took work home for homework. Comments that arose during the course of our sessions together were noted and changes were sometimes made. Once all the materials had been worked through the trainees attempted Diagnostic Assessment II since Diagnostic Assessment II was by then established as a parallel test to Diagnostic Assessment I. This meant that the assessments were used within around five weeks of each other.

Below is the table of pre-revision scores on Diagnostic Assessment I and post-revision scores on Diagnostic Assessment II.

Pre	Post	Gain
Revision		
%	%	%
48	60	12
22	58	36
16	50	34
44	70	26
14	50	36
34	72	38
42	68	26
46	70	24
22	62	40

From this table it can be seen that significant gains appear to have been achieved.

The trainees were asked to comment about the work they had been doing during the course of their revision and were also interviewed on their feelings about it. One very pertinent point was made: "You didn't make anyone feel like a dummy because they were experiencing difficulty with learning the subject."

One aspect of teaching basic mathematics to adults who have been "through the mill" before is that their confidence must be built up. To this end we need to be aware of the emotional as well as the conceptual domain of mathematics.

PRODUCTION AND RESULTS OF PILOTING THE AUDIO VISUAL MATERIALS

Considerable discussion took place about the content and style of presentation bearing in mind the particular needs of Skill Centre trainees. The production of such materials is costly. It was therefore decided to produce one programme and pilot it to evaluate how effective it was.

A programme "Multiplying Decimals" was produced and shown to a group of seven trainees who had been selected for revision but who had not received any revision at the time of seeing the programme.

At the end of the programme the trainees were given a questionnaire on the presentation, length, tempo, etc. of the programme. This information was supplemented by comments gained when the trainees were interviewed.

As a result it was decided that the most effective style of presentation was an informal one. The method of presentation was designed to encourage confidence in the trainees and to minimise any feeling of inadequacy with mathematics that so many of the trainees chosen for revision appear to experience.

Seven programmes were produced which were based on the first four units of the written materials (i.e. the core materials). All units, except unit 1, were split in two for the production of the programmes.

The Series is called "Working with Numbers" and the programme titles are: 1. Thinking of Numbers. 2. Multiplying Whole Numbers. 3. Dividing Whole Numbers. 4. Multiplying Decimals. 5. Dividing Decimals. 6. Adding and Subtracting Fractions. 7. Multiplying and Dividing Fractions.

IMPLICATIONS OF THE RESULTS

As a result of the study we have developed a set of parallel diagnostic assessments together with sets of complementary written teaching/self-learning packages and Audio Visual packages.

A small scale evaluation of the materials, made by the researcher, indicates that the tests used in conjunction with the written materials helps trainees revise and develop their basic mathematical concepts and skills.

The materials developed now need to be exposed to at least a small selection of Skill Centres to determine whether or not they can be used efficiently to help trainees revise basic mathematical concepts and skills. An evaluation study is clearly the next step.

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DRA-MATH

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INTRODUCTION

Dra-math is a new series of sixteen 25-minute video programs in elementary mathematics, produced by the Israel Instructional Television Center. The series serves as the backbone of the country's fourth grade math curriculum. The series consists of the following titles:

(1) Multiples and Divisibility.
(2) Why Shouldn't You Divide by Zero.
(3) Prime and Composite Numbers.
(4) Geometry 1.
(5) Factorization.
(6) Powers.
(7) The Decimal System.
(8) Geometry 2.
(9) The Arithmetical Operations.
(10) Divisibility Tests.
(11) Estimation and Rounding.
(12) Geometry 3.
(13) Long Division.
(14) Negative Numbers.
(15) Diagrams and Graphs.
(16) Geometry 4.

GOALS

The development of the series capitalized on knowledge gained from studies in five psychological aspects of mathematics education: motivation, math-anxiety, logical reasoning, communication and retention. Consequently, the series is aimed at achieving the following goals.

A. In the affective domain:
   (1) To increase motivation for studying mathematics.
   (2) To lower the level of fear of mathematics,

B. In the cognitive domain:
   (1) To provide for the development of mathematical-logical reasoning.
   (2) To encourage retention of basic concepts and principles.
   (3) To improve the assimilation of mathematical language into the natural speech of students.

The remainder of this paper shall analyze the way these goals are translated into operational terms. The last section contains details of a sample script.

MATHEMATICAL DRAMA

Drama is (assumed to be) a good tool for serving the goals in both the affective and the cognitive domains.

The programs present mathematics as an enjoyable experience through drama which,in most cases, serves as an integral part of the treatment of the mathematical topic.

The drama stems from surprising mathematical facts, patterns or puzzles. Typical errors, known to occur commonly, are treated in the programs by forming around them a conflict-situation, or by bringing them to an absurd point.

Here are a few examples of mathematical triggers of drama:
(1) The uniqueness of prime decomposition of a number is surprising, particularly in light of the numerous ways it can be decomposed when factors are not restricted to primes.
(2) The pattern of finding the quotient of an integer divided by a non-zero

integer, by changing the problem into a missing factor problem, becomes nonsensical when applied to zero as a divisor.

(3) Surprisingly, it is possible to know whether or not a number is divisible by 4 without knowing all its digits!

(4) What do we get by multiplying 2 by itself 16 times? Is it 2 x 16 or 2^{16}?

The broad potential of television is utilized to stimulate personal student involvement in the drama, and to turn mathematical challenges into intellectual recreation.

A WELCOMING ATMOSPHERE

Several decisions were made in order to create a friendly environment, thus aiming at the goals in the affective domain.

1) There is no one mathematical authority who teaches. A young actor whose hobby is math and excells in telling strange stories is the main adult character.
2) The cast includes children, allowing for identification of the audience with the screen action.
3) Girls do not play the traditional role of mathematical inferiors, and mathematics is not conveyed as a male role occupation.
4) Various segments of Israeli society are represented through the characters, and any stereotypical role of mathematical failure is eliminated. "I catch on too" is the desired classroom students' reaction to be gathered from viewing the programs.
5) Humor is not neglected as a way of enhancing familiarity. The same purpose is served also by mime-acting, magic tricks, science fiction, dreams, sports, outings, parties and games.

MATHEMATICAL EXPOSITION

With reference to the goals in the cognitive domain, the development of mathematical contents in the programs attempts to take into account the needs and limitations of the cognitive development level of the target population (ages 9 - 10) without relaxing the requirement of mathematical rigor.

The unfolding of the program content proceeds from concrete experience and the examination of particular cases, based upon analogies where appropriate, through intuitive generalization resulting from an analysis of examples and counter-examples, to finally observation of patterns and rules. As far as possible, new knowledge is not presented without accompanying explanation. The introduction of new subject matter is carefully based upon reasons anchored in logical arguments. However, formal proofs are not presented.

Mathematical language is kept as accurate as possible compromising only when it interferes with the natural communication ability of the target population (ages 9 - 10).

Short mathematical rhymes, often musical, provide means for retention of basic concepts and principles. They appear several times in each program, and subdivide it into a chain of short items.

ITEMARY STRUCTURE

Every program is composed of several items, each communicating a certain mathematical message. All items in one program treat the same topic, relating to various aspects of it, or to different levels of generalization in hierarchical order.

The itemary structure also enables audience attention renewal on one hand, and it facilitates diversity in production means on the other hand.

Let us now examine, for example, the itemary structure of one program: "Prime and Composite Numbers".

Production Style	Item Synopsis	Mathematical Content	Pedagogical Intention
1. Film. Real life situation. In the bakery.	Danny an actor, and his two young friends Limor and Dror, go to order a cake in a bakery. They help the baker's assistant to solve his problem: How many eggs should each cake contain if all cakes must be identical and the baker left him with 12 eggs. Luckily they find an extra egg.	12 has many (more than 2) divisors, while 13 has two.	An analysis of two particular cases, representing dichotomy which will be developed.
2. Studio. Danny's home.	Limor and Dror discuss the baker's problem. They pity the poor number 13 for having only two divisors. This leads Danny to distinguish prime numbers from composite numbers.	A natural prime number has two distinct divisors. A composite natural number has more than two.	Defining the concepts: Prime and composite numbers, with reference to the particular cases presented in the first item.
3. Animated musical rhymes.	Prime numbers - two divisors. Composite numbers - more than two.	as above	Memorizing a definition.
4. Studio. Theatrical situation.	Limor and Dror have fun putting on costumes. They tease one another investigating primality of 2, 3 and 4. The question of 1 rises	2 is prime, 3 is prime, 4 is not a prime. 1 is neither prime nor	Applying the definition to simple, particular cases,

Production Style	Item Synopsis	Mathematical Content	Pedagogical Intention
	naturally from their dialogue.	composite.	examples and counter examples. Examining an exceptional case.
5. Animated musical rhymes.	A repetition of #3 above.	see #3 above.	see #3 above.
6. A combination of film, studio and animation. Mime-acting with voice-over. Integrated with studio dialogues.	The baker's assistant sends the cake. The trouble he had with prime numbers reminds Danny of a story to tell the children about Tough and Luck who compete on an obstacle course numbered from 2 to 50. According to the rules they must not step on a composite number. Whoever does - suffers, and loses his turn. The winner is the one who gets safely to 50. Interwoven dialogues make Limor, Dror and the audience see the logic behind the winning strategy.	2 is prime. All other multiples of two are not. 3 is prime. All other multiples of 3 are not. 4 is composite. All multiples of 4 are multiples of 2. 5 is prime. All its other multiples are not. 6 is a multiple of 2 and 3. It is composite. 10, 15, 20 are multiples of 5 and either 2 or 3. 7 is prime. All its multiples are not. 14, 21, 28, 35 and 42 are multiples of 7 and of either 2, 3 or 5.	To build an explanation grounded in logical arguments for the sieve of Eratothenes.

Production Style	Item Synopsis	Mathematical Content	Pedagogical Intention
7. Studio. Danny's home	Dror is eager to eat the cake but Limor notices that Tough's and Luck's breaking points were: 4, 9, 25, 49. She discovers a relationship: $4 = 2 \times 2$; $9 = 3 \times 3$; $25 = 5 \times 5$; $49 = 7 \times 7$. Danny challenges the children about the next blow-up, if the obstacle-course was extended to 150.	Squared integers.	Examination of critical. Observation of a pattern. Application of a pattern for prediction.

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PME

Psychology of Mathematics Education

Proceedings of the seventh International Conference

Program
Additional papers

PME International Group for the Psychology of Mathematics Education
VII th CONFERENCE ISRAEL 1983
July 24th – July 29th

Program
Additional papers

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DEAR PARTICIPANT,

In this booklet you will find:

THE PROGRAM - which includes general information and the detailed program of the conference

ADDITIONAL PAPERS- papers which arrived late and could not be inserted in the Proceedings volume

LIST OF CONFERENCE PARTICIPANTS AND THEIR ADDRESSES

A conference is a result of a long process which demands the investment of many kinds of effort:

- time, money and work.

I want to thank all those who were involved in this process:

The conference committee members
The various funds and the higher education institutions in Israel which by their support demonstrated their interest in the subject of *Psychology of Mathematics Education*.

In addition to the above, I want to stress the work and help which were given in good spirit by my colleagues in the Science Teaching department.

Finally, we hope that you will enjoy the conference and benefit from its formal and informal scientific program.

Best wishes,

Rina Hershkowitz.

Program

GENERAL INFORMATION

PLENARY LECTURES	will be held in lecture room 1.
WORKING SESSIONS	will be held in lecture rooms 1, 2, 3 and 4. For any queries concerning the program, please contact TOMMY DREYFUS.
POSTERS AND SMALL EXHIBITS	will be displayed in lecture room 5, from Monday through Thursday. Participants are asked to contact DAVID BEN-HAIM for technical details. On Wednesday afternoon, from 18.00 to 19.30, time has been allocated for the discussion of posters and small exhibits. Contributors are requested to be present at their displays during this time.
CONFERENCE SECRETARIAT	will be open daily from 8.00 to 15.30 in the foyer of the Guest House.
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MEALS	All meals will be served in the main dining hall of the Guest House.

BREAKFAST	From 7 o'clock a.m.
LUNCH	12.45 p.m.
DINNER	See the program for the time.

IDENTIFICATION BADGES	Participants are asked to wear their badges at all times during the conference.

PROCEEDINGS	will be included in the participant's kit. Extra copies of the Proceedings can be purchased at a cost of $ 10 at the conference secretariat.
PME MEMBERSHIP	is a necessary condition for attending the Conference. Your membership fee can be paid upon arrival at the Guest House or at the conference secretariat during the Conference.

OVERALL PROGRAM SCHEDULE

Day	8.30	9.45	10.15	12.45	15.00	15.30	18.00
Sunday July 24				*Check-in at hotel and registration*			*Dinner (19.30-22.00)*
Monday July 25	*Opening & 1st plenary lecture*	*Coffee break*	*Working Sessions*	*Lunch and free time*	*Coffee*	*Working Sessions*	*Free time* *Dinner (19.00)* *Dance evening (20.30)*
Tuesday July 26	*2nd plenary lecture*	*Coffee break*	*Working Sessions*	*Lunch*	*Tour of Jerusalem*		*Dinner and concert in Jerusalem (19.00)*
Wednesday July 27	*3rd plenary lecture*	*Coffee break*	*Working Sessions*	*Picnic and tour of stalactite cave*			*Coffee* *Working Sessions (18.30)* *Dinner (20.15)* *AGM (21.00)*
Thursday July 28	*4th plenary lecture*	*Coffee break*	*Working Sessions*	*Lunch and free time*	*Coffee*	*Working Sessions*	*Free time* *Barbecue + Israeli folklore & Dancing*
Friday July 29	*Working Sessions*	*Coffee break (10.50)*	*Closing plenary lecture (11.15)*	*Lunch and departure*			

MORNING MONDAY, JULY 25	ROOM 1	ROOM 2	ROOM 3	ROOM 4
OPENING 8.30 PLENARY LECTURE	*Greetings: P. Nesher* *E. Fischbein: Role of implicit models in solving elementary arithmetic problems* p.2 (AMOS DE SHALIT MEMORIAL LECTURE)			
BREAK 9.45				
10.15	*C. Comiti* p.146 *"Who's got the highest number?"*		*E. Cohors-Fresenborg* *Cognitive styles* p.333 *in the field of computer programming*	
WORKING	**Travel time**		**Travel time**	
SESSIONS 10.40	**Discussion**	*M.C. Mitchelmore* * *3d drawings and the parallel concept*	**Discussion**	*R. Rees* * *Studies of mathematical ability - some highlights*
		Travel time		**Travel time**
(20 min. presentation 30 min. discussion) 11.05	*N. Francis* p.153 *Special difficulties of Arab pupils with numbers, ...*	**Discussion**	*J.M. Moser* p.339 *Using a microcomputer to teach representational skills*	**Discussion**
	Travel time		**Travel time**	
11.30	**Discussion**	*D. Ben-Haim* p.235 *Spatial Visualization Sex and grade level differences in grades ...*	**Discussion**	*A. Vermandel* *Mathematical thinking and mathematical structures*
		Travel time		**Travel time**
11.55	*J.F. van den Brink* p.158 *Numbers in contextual frameworks*	**Discussion**	*U. Leron* p.346 *Some problems in children's logo learning*	**Discussion**
	Discussion		**Discussion**	

* See additional papers

AFTERNOON MONDAY, JULY 25		ROOM 1	ROOM 2	ROOM 3	ROOM 4
	12.45	LUNCH + FREE TIME + COFFEE (15.00)			
	15.30	*A. Bell p.205* *Diagnostic teaching of additive and multiplicative problems*		*K.F. Collis p.68* *Cycles of learning and the school mathematics curriculum*	
		Travel time		**Travel time**	
WORKING SESSIONS	15.55	**Discussion**	*K. Hasemann p.229* *Analysis of children's discussions of geometrical problems with ...*	**Discussion**	*M. Reiss p. 421* *Does the grouping of students make a difference – on the ...*
			Travel time		**Travel time**
(20 min. presentation 30 min. discussion)	16.20	*B. Greer ** *Understanding multiplication and division – context, ...*	**Discussion**	*H. Meissner p.76* *How to prove relational understanding*	**Discussion**
		Travel time		**Travel time**	
	16.45	**Discussion**	*T. Dreyfus & T. Eisenberg p.241* *Spatial reasoning: Stages of development*	**Discussion**	*F. Friederwitzer & B. Berman p. 408* *Mathematics inservice that works: ...*
			Travel time		**Travel time**
	17.10	*M. Swan p.211* *Teaching decimal place value – a comparative study of "conflict" and "positive only" approaches*	**Discussion**	*K. Crawford p.95* *Past experience and mathematical problem solving*	**Discussion**
		Discussion		**Discussion**	

EVENING EVENTS	19.00	DINNER
	20.30	DANCE (ISRAELI AND OTHERS)

* see additional papers

MORNING TUESDAY, JULY 26	ROOM 1	ROOM 2	ROOM 3	ROOM 4
PLENARY LECTURE 8.30	*K. Hart: Tell me what you are doing - discussions with teachers and children p.19*			
BREAK 9.45				
10.15	*N. Bednarz & M. Belanger Problems of representation of an operation in ... p.188*	*SYMPOSIUM ON INTUITION AS APPLIED TO THE NOTION OF FUNCTION* *Discussion leader:* *E. Fischbein*	*W.K. Ransley p.107 "Real life" numeracy arithmetical competence and prediction from ...*	
	Travel time		**Travel time**	
WORKING SESSIONS 10.40	**Discussion**		**Discussion**	*G. Heink & C. Wichrowski Mathematical models - a helpful instrument for ... p.336*
				Travel time
(20 min. presentation 11.05	*N. Herscovics, J.C. Bergeron & C. Kieran A critic of Piaget's analysis of ... p.193*		*S. Yeshurun p.120 Practical applications of psychomathematics and neuropsychomath...*	**Discussion**
	Travel time			
30 min. discussion) 11.30	**Discussion**		**Discussion**	*J. van Dormolen p.391 On the correctness of mathematical content in texts*
				Travel time
11.55	*J.C. Bergeron & N. Herscovics p.199 Models of multiplication based on the concept ...*		*U. Fidelman p.114 Learning of non-standard arithmetic and the hemispheres ...*	**Discussion**
	Discussion		**Discussion**	

AFTERNOON AND EVENING

TUESDAY, JULY 26

12.45 LUNCH

13.45 TOUR OF JERUSALEM

19.00 DINNER - *HEBREW UNIVERSITY, MOUNT SCOPUS, THE MAIERSDORF HOUSE.*

20.30 CONCERT - *HEBREW UNIVERSITY, MOUNT SCOPUS*

MORNING *WEDNESDAY, JULY 27*		ROOM 1	ROOM 2	ROOM 3	ROOM 4
PLENARY LECTURE	8.30	*H. Freudenthal: Is heuristics a singular or a plural? p.38*			
BREAK	9.45				
WORKING SESSIONS (20 min. presentation 30 min. discussion)	10.15	*B. Southwell p.170* *The development of rational number concepts*		*D. Pimm p.52* *Against generalization: Mathematics, students and ulterior motives*	
		Travel time		**Travel time**	
	10.40	**Discussion**	*N.D. Fisher p.218* *Some perceptual influences in learning geometry*	**Discussion**	*F. Lowenthal p.364* *Strategy games, winning strategies and non-verbal ...*
			Travel time		**Travel time**
	11.05	*L. Streefland p.182* *The long term learning process of ratio*	**Discussion**	*W. Dörfler p.57* *Models of the process of mathematical generalization*	**Discussion**
		Travel time		**Travel time**	
	11.30	**Discussion**	*R. Hershkowitz & S. Vinner p.223* *The role of critical and non critical ...*	**Discussion**	*A. Floyd p.369* *Children's learning strategies in mathematics ...*
			Travel time		**Travel time**
	11.55	*C. Pinchback* *Computational errors of students in grades 4-8*	**Discussion**	*J. van Geel, E. Schillemans & A. Vermandel p.63* *About the mental action ...*	**Discussion**
		Discussion		**Discussion**	

AFTERNOON WEDNESDAY, JULY 27	ROOM 1	ROOM 2	ROOM 3	ROOM 5
12.45	PICNIC + TOUR OF STALACTITE CAVE + COFFEE (18.00)			
18.30		A. Abele p.278 Intuition and learning of the function concept	E. Schillemans p.101 Mathematics and self-actualization	PRESENTATION OF POSTERS AND SMALL EXHIBITS - QUESTIONS AND ANSWERS WITH AUTHORS OF POSTERS AND SMALL EXHIBITS
		Travel time		
	N. Movshovitz-Hadar & T. Reiner p.436 Dra-math	Discussion	Discussion	
		Travel	time	
19.20	Discussion	A. Kreimer & N. Taizi p.284 Graphic and algebraic presentation of functions - can ...	G. Ervynck p.426 Experiences concerning mathematical education in Africa	
		Discussion	Discussion	
EVENING EVENTS	20.15 DINNER 21.00 AGM			

MORNING *THURSDAY, JULY 28*		ROOM 1	ROOM 2	ROOM 3	ROOM 4
PLENARY LECTURE	8.30	*D. Wheeler: Some problems that research in mathematics education should address*			
BREAK	9.45				
WORKING SESSIONS (20 min. presentation 30 min. discussion)	10.15		*G. Holland p.259* *Working backwards in solving geometric calculation problems*	*H. Rin p.295* *Linguistic barriers to students' understanding of definitions*	
			Travel time	**Travel time**	
	10.40	*I. Wachsmuth, M.J. Behr & T.R. Post p.164* *Children's perception of fractions and ratio in ...*	**Discussion**	**Discussion**	*H. Osser p.82* *Forms of knowledge in the child's learning of mathematics*
		Travel time			**Travel time**
	11.05	**Discussion**	*G. Becker p.247* *Can we teach heuristic strategies?*	*S. Vinner p.289* *The notion of proof - some aspects of students' views at ...*	**Discussion**
			Travel time	**Travel time**	
	11.30	*R.P. Hunting p.176* *How children account for fractions equivalence*	**Discussion**	**Discussion**	*J. Adda p.88* *Representations in mathematics*
		Travel time			**Travel time**
	11.55	**Discussion**	*N. Hadas, T. Dreyfus & A. Friedlander p.253* *Euclidean geometry for average ability ...*	*Z. Mevarech & Y. Dosti* *Students' p.313 misconceptions of the equivalence ...*	**Discussion**
			Discussion	**Discussion**	

AFTERNOON *THURSDAY, JULY 28*	ROOM 1	ROOM 2	ROOM 3	ROOM 4
12.45	LUNCH + FREE TIME + COFFEE (15.00)			
15.30		*N. Zehavi & M. Bruckheimer* p.395 *A case study of teachers' thinking ...*	*L.R. Booth* p.307 *A diagnostic teaching programme in elementary algebra: results and ...*	
		Travel time	**Travel time**	
15.55 WORKING SESSIONS	*J.F. Richard & J.C. Escarabajal* p.139 *Comprehension and solution processes in word problem ...*	**Discussion**	**Discussion**	*J. Hillel* p.38 *A task-oriented method of protocol analysis*
	Travel time			**Travel time**
16.20 (20 min. presentation 30 min. discussion)	**Discussion**	*C. Hoyles* p.402 *The mathematics teaching project (T.M.T.P.)*	*I. Kidron & S. Vinner* p.301 *Rational numbers and decimals at the ...*	**Discussion**
		Travel time	**Travel time**	
16.45	*M.L. Leite Lopes* p.142 *Problem solving at elementary school*	**Discussion**	**Discussion**	*S. Pirie & J. Tuson* p.376 *A new approach to mathematics testing*
	Travel time			**Travel time**
17.10	**Discussion**	*G. Barr* p.430 *Research and curriculum development: the development of ...*	*R. Falk* p.319 *Experimental models for resolving probabilistic ambiguities*	**Discussion**
		Discussion	**Discussion**	
EVENING EVENTS	FAREWELL BARBECUE ISRAEL FOLKLORE EVENING + DANCING			

MORNING FRIDAY, JULY 29		ROOM 1	ROOM 2	ROOM 3	ROOM 4
	8.30	*E. Teubal & P. Nesher p.124* *Order of mention vs. order of events as determining factors ...*		*C. Janvier p.266* *Representation and understanding: the notion of function ...*	
		Travel time		**Travel time**	
	8.55	**Discussion**	*D. Tall & G. Sheath p.357* *Visualizing higher level mathematical concepts ...*	**Discussion**	
			Travel time		
	9.20	*G. Booker* * *Children's construction of a number system*	**Discussion**	*Z. Markovits, B. Eylon & M. Bruckheimer p.271* *Functions - linearity unconstrained*	
		Travel time		**Travel time**	
	9.45	**Discussion**	*A. Vidal-Madjar p.352* *Learning statistics with the help of the computer*	**Discussion**	
			Travel time		
	10.10	*E. Diez-Martinez & A. Guerrero p.136* *Young children's comprehension of "more" and ...*	**Discussion**	*R. Even & A. Kreimer p.326* *Problem solving: a correspondence course*	
		Discussion		**Discussion**	
BREAK	10.50				
CLOSING PLENARY LECTURE	11.15	*G. VERGNAUD: Let us discuss theory and methodology*			
	12.45	LUNCH			

* See additional papers

Additional papers

PLENARY PAPER

SOME PROBLEMS THAT RESEARCH IN MATHEMATICS EDUCATION SHOULD ADDRESS

David Wheeler
Concordia University, Montreal

INTRODUCTION

This paper is based on the results of a small informal enquiry into the research problems judged to be significant for the future development of mathematics education. Before I embark on the main part of the paper, I want to put the question: *What do we expect teachers of mathematics to gain from the research that we do?*

It is sometimes said that mathematics education research has no effect on the teaching of mathematics in schools. It is perhaps truer to say that it has an unpredictable effect, and possibly truer still to say, with hindsight, that its effect is greatest when the research is at its worst, at its most simplistic and reductionist.

What use could teachers make of research anyway? By its nature, research is concerned to make statements of general validity, whereas the teacher's problems are always specific and frequently idiosyncratic.

The argument can be countered fairly readily, of course. But a more difficult question concerns the mechanism by which teachers can find out what research has to tell them. I won't rehearse all the familiar stuff about better communication, but will say only that efforts to summarise and digest research results for teachers (or anyone else) are dangerous in the extreme. If, as is usually the case, such summaries omit all the evidence on which the researches were based, they implicitly invite teachers to believe what they are told for no better reason than that someone has thought fit to tell them.

This not only treats teachers in the way we do not want them to treat their students, it also trivialises the research activity. We have to develop more complex ways of dealing with the communication of research.

But to return to my question. Let me suggest four possible benefits that teachers could derive from research.

(1) It raises consciousness, particularly when it surprises. It can tell us things we didn't suspect, and attract our attention to items of experience that have been overlooked.

(2) It adds to the teacher's technical resources. Researchers, unfortunately, are not as interested in this outcome as I think they should be. Standardised tests are not the only kind of resource that research can lead to. Ideas, too, are tools.

(3) It gives the teacher a theoretical framwork that works better than the folklore of the classroom. Theories, as Polanyi points out, are more objective than experiences.

(4) It demonstrates that there are ways of dealing with "why?" even in a complex, elusive situation like the classroom. It shows teachers that one can learn how to set about answering one's own questions.

Although intellectual curiosity has the best motive power and staying power for research, in an applied field like mathematics education it is neccessary for research to draw from, and feedback into, the places where mathematics education takes place. If we neglect what teachers need and what they could use, we must call our research something other than research in mathematics education.

THE ENQUIRY

In January of this year I sent a letter to 60-plus mathematics educators asking them to nominate specific problems in mathematics education, drawing the parallel with Hilbert's famous address to the 1900 IMU Congress in which he proposed 23 problems whose solution would advance mathematics in the following decades. Although aware that this was not a new idea (I remembered occasional discussion with colleagues in ATM), I didn't quite realise that I was treading hard on the heels of both the ICME-4 organisers, who proposed this exercise to Freudenthal for his address, and of the editor of the NCTM Research in mathematics education volume, who proposed it to his contributors. I imagine that there is some significance in these (and no doubt other) independent arrivals at the same question. Perhaps a number of people working in mathematics education would like the reassurance that their research does have a direction, that they may look ahead to what may be attainable in the future. They can see small signs of progress on some fronts, giving them the feeling that some evolution may be under way, so it is not totally crazy to ask the question, though no one believes (I suggest) that mathematics education is presently advanced enough, or coherent enough, to endow Hilbert-like insights with anything like the same prognostic power.

For myself it was a way to raise the question of the contribution of research to mathematics education. Do we know what research in this field can do? might do? can't do? shouldn't attempt to do? People undertake research for all sorts of reasons, not all of them associated with a burning desire to know more and understand better; they choose their research area from those which are favoured, fashionable, convenient and foolproof as well as from those which are crucial and productive. These are facts of the research milieu that we all know and need not be too high-minded to admit. But an organisation like PME, for example, has some sort of duty, it seems to me, to undertake some monitoring, some review of priorities, and some consciousness-raising, about the overall research enterprise.

I received 21 replies to the letter, 4 of which were group efforts. (The names of the principal respondents are listed in the notes at the end of the paper.) I am, as you might expect, in some difficulty in collating the contents of even this small number of replies and can only give a fairly broad impression here. Many of the replies are worth considering individually and in detail, and I hope to be able to publish all the replies with some commentary in For the Learning of Mathematics or elsewhere.

Some respondents (less than I had expected, but perhaps the 40-plus who didn't reply would swell the proportion) referred to the intrinsic difficulty of arriving at solvable problems in mathematics education (Balacheff, Bishop, Brown, Carpenter, Howson, Kilpatrick, Mason). There is not only the difficulty of formulating problems in a field lacking a well-articulated theoretical framework (Balacheff) but also the difficulty of arriving at anything that could count as a solution. Perhaps solutions in mathematics education are quite unlike solutions in other fields (Carpenter), and how do we recognise a solution when we have reached it (Tahta)? Perhaps solutions are necessarily limited and local (Howson), perhaps they have to be reworked in each generation (Kilpatrick). The field contains issues and puzzlements rather than problems in the Hilbertian sense (Brown et al) and we plunge into "pools of ignorance" (Bishop). What mathematics education needs is not answers but actions (Mason).

Even fewer respondents referred explicitly to the difficulties that follow mathematics education research, particularly the daunting difficulty of raising it to the status of information that could make a difference to the teaching of mathematics. How do we reach teachers (Skemp)? Is our research strategy suited to the task of making an impact on teaching (Fischbein)?

What is mathematics education anyway, and what are its boundaries? References to "a domain of personal growth and interaction" (Mason) and "being dishonest if we look at mathematics and believe that it doesn't touch our intricacies of being alive and alert" (Brown et al) seem to be claiming larger tracts of territory than do the research accounts that numerically dominate the literature. (And here I am triggered to remind you that one of Freudenthal's "problems", not referred to by any of my respondents, is the difficulty of picking out the good research from the junk.) Certainly the researcher's view of mathematics education will strongly influence the questions he or she will try to answer about it. Perhaps we should clarify the basic, general objectives of mathematics education (Fischbein).

In attempting some classification of the replies, I first thought of adopting the triad "la mathématique, l'élève, la situation scolaire" (Balacheff), locating the suggestions with reference to these points and their joins. This fits a number of the problems with a didactic or pedagogical orientation fairly well, but it excludes problems of a more general or more theoretical nature as well as problems arising from the research activity itself. In the end, after reviewing other schematic sortings, I decided to take an ad hoc approach to the data for this paper.

Problems on the table. You will not be surprised that computers (Bishop, Fischbein, Mason, Robitaille et al, Tahta, Tall, Vergnaud, Wachsmuth et al), problem solving (Balacheff, Brown et al, Fischbein, Robitaille et al), and the differential mathematical attainments of e.g. women and blacks (Barbeau, Confrey), figure among the suggestions. Very different in character as they are, the three have in common that they are the focus of attention (or at least the object of some concern) not only of researchers; they exist as already identified problems for teachers. I will not discuss these further except to say that their "topicality" should give research a chance of having an impact on what happens in schools.

Problems in the air. Some areas of research appear to be developing into major centres of interest, though they do not have immediate prospects of applicability like those in the first category. I code them simply as language, imagery and the brain.

> Are there different levels of use of oral and symbolic language in the personal construction of mathematics (Kieren)? What are the obstacles that learners find in formal and symbolic mathematical language (Wachsmuth et al)? How can children acquire mathematical symbolism and language as naturally as they acquire their native language (Agassi)? What are the differences between learning from a text and learning from a person (Bishop)? How is mathematical meaning communicated (Bishop)?
>
> What pictures of mathematical ideas do students have in their minds (Barbeau)? Are there such things as canonical images in mathematics (Tahta)? What are the effects of, say, "Turtle" imagery or the imagery of set theory (Kuyk)? What image(s) of mathematics do teachers have (Confrey)?
>
> What can a neurological and physiological study of brain activity tell us about mathematical activity (Kuyk)? Can we explain mathematical ability neurologically (Howson)? If everyone has the inborn capacity to become an excellent calculator, why does the skill show up so rarely (Agassi)? Is the computer a good model of the brain (Tall)?

(I note here that throughout the paper I am condensing and paraphrasing respondents' statements, so not much weight should be given to the precise formulations.)

Problems from mathematics. Three references are made to axioms: Kieren conjectures that mathematical axioms point to the ingredients required to build personal mathematical knowledge, Kuyk asks if the mathematician's need to prove his knowledge within an axiomatic structure can be reconciled with the student's need to acquire his knowledge from situations allowing a high level of intuitive response, and Mason wonders what personal experiences are presupposed by the axiomatic construction of arithmetic.

Two references are made to history: Brown asks whether regarding mathematics as an activity rather than a body of knowledge impoverishes it by depriving it of a sense of history, and Tahta wonders what pedagogical message follows from the observation that as it develops mathematics reconstructs its own history.

Two references are made to "new math": Kuyk conjectures that "new math" by emphasizing computability, algebra, linearity, all requiring a low level of mental concentration, has tended to displace mathematics that requires imagination and a high level of ingenuity; Tall asks what happened to "new math" and what did we learn from it.

Barbeau calls for more attention to the various modes of mathematical thinking and the connections between them. Fischbein proposes problems about mathematical intuition and mathematical proof. Tahta suggests an epistemological study of elementary algebra and elementary geometry.

It may be significant that most of the problems concerning mathematics are raised by respondents with strong mathematical interests. Is mathematics plain sailing for everyone else, or is it that the rest of us don't know what questions to ask?

Problems from pedagogy. This is a small category in my classification, mainly because I have put problems elsewhere when I could. But I am left with a few that have nowhere else to go.

For example, Kilpatrick and Kuyk are concerned about the hierarchical model of presenting mathematics and wonder how far it corresponds to necessary conceptual hierarchies within mathematics itself. Bauersfeld and Bishop ask for more investigation of the relation between mathematics and experience of the real world so that "embodiments" may be more effective and classroom problems more realistic as well as more mathematically credible.

Problems from research. The potential contribution of cognitive science to mathematics education is acknowledged in several contributions (Bauersfeld, Greeno et al, Tall, Wachsmuth et al). It has produced concepts such as "debugging", "frame", etc. and models of different intellectual functionings that could be applied in mathematics education. Only Greeno et al puts an entire research programme in a cognitive science setting.

Some problems seem to belong here because they have been around for ever. What is good teaching (a good teacher)? is one (Tahta). Transfer is another (Bauersfeld, Greeno et al, Kilpatrick) and mathematical ability/aptitude another (Fischbein, Howson). I am rather pleased that Bishop suggests the banishment of the concept of mathematical ability (on the grounds of the harm its misuse has caused), and I would propose the prompt extinction of the word "transfer" as I am convinced that the word itself is the problem, not the phenomenon to which it is supposed to refer, and which may or may not exist.

It may be appropriate to insert here the remark that theoretical frameworks, and the concepts they enshrine, can be obstacles as well as aids. We are not, I hope, so desperate for a conceptual framework that we will pick up and use any words that are lying around. It is rather easy to fall into supposing that if we have a word for something, that something must exist. The opposite situation - that we don't take account of something that exists because we don't have a word that denotes it - is less of a trap, but an inhibition nevertheless. Even when there is a word there may be a whole range of things to which it refers, not all equivalent to each other. The word "problem" itself, while indispensable, doesn't carry the same meaning for everyone, as Glaeser's review in a recent issue of ESM demonstrates. There are, one hopes, family resemblances between the meanings attributed to such a word (as there is to "number"). When there aren't, as seems to me to be the case, for example, with "learning style", then the concept is too unstable to be of any use.

An emergent problem. In scanning the 21 papers I received, it occurred to me that a number of proposals concerned change. Vergnaud proposes a study of the ways children's conceptions of arithmetical operations change. Confrey asks how children's misconceptions change into correct conceptions, and how students change into teachers during a professional training programme. Robitaille et al and Confrey refer to the problem of curriculum change, and the changes required of teachers in implementing curriculum change. Wachsmuth et al asks how conputation and proof change with the advent of computers. Mason asks what makes people's images of mathematics change. Many others could be put into this category without much distortion, even in those cases where change is not specifically referred to.

If change, actual or desired, is a prominent feature of many research problems, perhaps one ought to ask if there are research methodologies particularly suitable for dealing with this variable.

SOME COMMENTS ON THE ENQUIRY

This subheading is probably misleading. It does not seem to me wise to make too much in the way of comment on the previous section, but let it speak for itself. You know that the contributions are from a small, biased sample; you cannot know how cavalier I have been in my selection of items to quote, or how savage in my condensations. But it is clearly frail evidence to generalise from. Let me raise some questions that may expand the context of the discussion a little.

The first question is triggered by that sense of déjà vu that all of us must experience at times. When I read, say, Ginsburg and Seeley Brown's researches that show that children's computational errors are not random or stupid but the rational consequences of certain misconceptions, I say to myself that I knew that 30 years ago soon after I began to teach. We find in Brownell, and no doubt in earlier researches, that this was also known long before I knew it. Is Kilpatrick right in suggesting that we have to re-invent the wheel in each generation? Perhaps we should do some research to find out why knowledge that could have been circulated through the profession was not, or was circulated and was not believed. That might be a better way of spending research time than, in twenty years' time, making the same discovery again.

I do not know how many of you may feel, with me, that the above discovery is not in the least surprising. Who could think that children would not be sensible and rational, given the choice? Once one has known a few children, it would offend against commonsense to suppose they would not act rationally. But a lot of what happens and is believed in classrooms, it seems to me, offends against commonsense. One wonders how teachers can possibly do some of the things they do, or believe some of the things they say they believe. (One wonders also, I hasten to add, about the offences against commonsense committed by not a few researchers. I don't hold teachers to be more blind than other groups.) Do teachers, like Ginsburg's and Seeley Brown's students, act rationally on the basis of their misconceptions, or is the matter more complex than this? Does research stand any chance of reinstating commonsense in this situation? I doubt if it is sufficient, or even if it is the right tool. Certainly we cannot expect research to solve all the mental health problems of mathematics teaching, and it doesn't help to talk as if it could.

There is probably too much research. There is, I believe, far too much collecting of data, almost a worshipping of data, though data without point and without interpretation are just empty marks on paper or empty remarks on tape. Empirical observations serve only to trigger thoughts and explanations, and to check the more extravagant flights of both. That's an important function, indeed, but a lot of good research is non-empirical, or only empirical at the fringes. Yet the overwhelming proportion of published research (in North America, at least) servilely follows "scientific" paradigms of observational or experimental study, as if these were the only research plans that could lead anywhere. Research plans, anyway, are largely a nonsense. A research problem honestly adopted and lived with will dictate the steps of its solution as it is worked on.

CONCLUSION

Where has this "Hilbertian game" taken us? We can indeed formulate problems in mathematics education, significant problems, and it would not be difficult to find 23 of them. But the similarities seem to end there. Will we know when we have solved them? Will the solutions be valid for other times and other places? Will the solutions be integrated into the field so that they become part of the store of knowledge for future workers? The answer to all three questions, as far as one can tell at the moment, is likely to be "No".

I do not think we should expect or even hope that the answer will ever be "Yes". The kind of knowledge we seek in mathematics education is not absolute knowledge - if such a kind of knowledge about anything can exist - but what I will call (until I can think of a better term) "cultural knowledge". By this I mean "what a culture knows", a fusion of cultural and objective understanding similar to the fusion of personal and objective understanding described by Polanyi (1958) as personal knowledge. Education, and mathematics education in particular, is a social enterprise and we forfeit the possibility of making sense of it if we ignore the values and intentions of the society that provides it (Keitel, 1982).

I am airing this concept for the first time (in my own writing, I mean) and have not properly begun to work it out to see if it is more than an interesting intuition. Through the haze surrounding it I believe I can get a glimpse of the reason why our research efforts have so little impact on the classrooms in which our education is largely carried on. Personal knowledge, as Polanyi reminds us, requires a personal commitment. The possibility of cultural knowledge depends on cultures committing themselves to rational processes of knowing and deepening understanding. Unfortunately for us, we seem to be in the wrong places at the wrong time; we beaver away, but no one seems to want to know.

It is not clear that this way of expressing our dilemma is of any help, but the fact that our best research efforts have no discernable effect on the education that our children receive remains the single problem most urgent and important for us to solve.

June 30, 1983

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UNDERSTANDING MULTIPLICATION AND DIVISION: CONTEXT, PROBLEM STRUCTURE, NUMERICAL EFFECTS AND THEIR INTERACTIONS

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In this paper I will outline some (by no means all) of the factors affecting comprehension of situations which can be modelled by the multiplication or division of two positive (real) numbers. Various exploratory investigations (using pencil-and-paper tests, clinical interviews, and tryouts of teaching materials) will be used for illustrations, the main source being Bell, Fischbein and Greer (1983).

The use of verbal problems for which the operation has to be stated goes back at least as far as Courtis (cited in Thorndike, 1926) who used such items as a test of reasoning in arithmetic. Some points of interest are his conclusion that the test mainly measured reading ability; the reported "difficulties experienced by teachers in attempting to remedy the defects in reasoning"; and the finding that the difficulty of an item could be markedly changed by rephrasing.

FACTORS AFFECTING PERFORMANCE

For the purposes of this paper, I will concentrate on three groups of factors, which interact in numerous ways.

1 Effects associated with the numbers involved: Certain misconceptions are well-known (Hart et al, 1980; Bell, Swan and Taylor, 1981) in particular the invalid generalizations from positive integers that "multiplication makes bigger, division smaller" and "larger always divided by smaller".

2 Effects of problem structure: The differences between partitive and quotitive division has long been recognized; in fact, different algorithms used to be taught for the two cases. We have used the term Fractional Partition for situations such as "4 pies divided among 12 children" and Fractional Quotition for situations such as "£60 is needed to buy a bicycle and £40 has been saved - how much of the amount needed is this?". Situations such as price, speed etc we classify as Rate structures, and other division structures we distinguish are Measure Conversion, and Enlargement. In multiplication we distinguish between situations reducible to Repeated

Addition (a) of numbers, (b) of measures; Rate structures; Measure Conversion and Enlargement. (This classification is not exhaustive; we have not included Cartesian product, for example.)

3 Effects of specific contexts: there is further variation associated with different contexts, due to such factors as familiarity, and amenability to representation by diagram or image. For example, the question "How long does it take to go 38 miles at 7 miles per hour?" is presumably both more familiar and more amenable to representation by a line diagram, than is "What is the volume of 38 gms of a material of density 7 gms per cc?" though both have the same structure in terms of our classification.

A further set of factors which are not considered here, but which are certainly important, are those associated with the precise wording of the problem.

SELECTED EXAMPLES OF FINDINGS

1 Accessibility of division structures

One of the techniques adopted by Bell et al (1983) was to give pupils calculations and ask them to make up stories for which these calculations were appropriate. Here I will consider the results for division items. The only structures used by the pupils were Partition (P), Fractional Partition (FP), Quotition (Q), Fractional Quotition (FQ) and Rate (R). The 8 items fell naturally into four levels, in terms of which of these structures they fit, as follows:

Level 1	(18 ÷ 3, 12.7 ÷ 9)	P, Q, R
Level 2	(4 ÷ 24, 2.71 ÷ 18)	FP, FQ, R
Level 3	(15.2 ÷ 2.4, 0.74 ÷ 0.21)	Q, R
Level 4	(8.7 ÷ 59.1, 0.47 ÷ 1.3)	FQ, R

At Level 1, 82% of responses were correct, of which almost all were of type P. At Level 2, 65% were correct, of which almost all were of type FP (on the first item 40% of the responses were of type P, the order of division having been reversed). At Level 3, 52% of responses were correct, of which almost all were of type Q. Only 12% of responses at Level 4 were correct, of which three-quarters were of type R. It appears, therefore, that Partition and Quotition (though not Fractional Quotition) structures are easily accessible, the former being consistently preferred when both are possible. Rate embodiments were much less common (and many attempts to use rate structures led to errors). Many errors represented attempts to distort the calculation given to make it fit a more accessible structure (i) by altering the numbers in some way, such as by changing units (ii) by reversing the order, or (iii) by "forcing" the calculation into a structure it did not fit. A follow-up experiment is needed in which the children would be asked to write P, Q and R stories (and perhaps some others) for each calculation.

2 Precise conditions for numerical misconceptions

(a) "Multiplication makes bigger, division smaller": A number of findings point to the generalization that this misconception comes into play whenever the solution process for a o b = c leads the pupil to consider the relative sizes of a and c, but not otherwise. For example, take the item "The petrol tank of a Mini holds 5.5 gallons. How much is this in litres? (One litre is 0.22 gallons)". In this case, the comparison between the number of gallons required to fill the tank, and the larger number of litres, is salient. Of the 30 pupils tested, 12 wrote 5.5 x 0.22 as the answer. Questions involving scaling by a factor less than one provide another example where the contrast is salient, and division is very often given as the operation in these cases. On the other hand, for Quotition items (eg 17.6 m of rope is cut into pieces 0.7 m long; how many pieces will be obtained?) a comparison between 17.6 and the answer is less likely, and multiplication is rarely given as the answer in these cases.

(b) "Larger always divided by smaller" In an interview with one girl, discussing the question "A race 4.8 miles long was run by the winner at an average speed of 5.24 miles per hour. How long did it take him?" she said she would divide 4.8 into 5.24 "because it was the smaller number ... it would be easier to do". We have encountered numerous variations on the general problem of the order of the numbers in division, and the misconception that the larger number is always divided by the smaller is only part of the story. For example, for the item "How many cushion covers needing 0.48 m each can be made from 2.4 m of material?" 12 of the 30 pupils answered 0.48 ÷ 24, presumably distracted by the salience of 24 dividing neatly into 48. In future investigations we hope to tease out the various factors which affect the choice of order, given that the operation has been identified correctly as division.

3 Non-conservation of operation

Within the domain we are considering, it is taken for granted by us that the operation which models a given situation is the same regardless of the numbers involved. For pupils, however, this is not the case, and we have frequently observed cases where the pupils believed that the operation changed when the numbers were changed. Some pupils we interviewed did not perceive any conflict in using multiplication to work out the cost of 2 gallons of petrol at £1.33 per gallon, and division to work out the cost of 0.53 gallons. This has implications for the effectiveness of the strategy of working out the operation by replacing the numbers in the problem by easy numbers.

IDEAS FOR FURTHER INVESTIGATIONS

1 One way of appreciating the invariance of an operation (say to work out the cost of petrol given price per gallon and number of gallons) is to write a computer program which gives the answer when fed the data. An actual computer is not necessary; a pupil can be put in a box with a calculator with the instructions that whenever two numbers on paper come through one hole in the side he must multiply them on the calculator, write the answer on a piece of paper, and put it out through another hole.

2 Microcomputers offer the chance to represent functional relationships dynamically. For example, we are trying out a program which represents buying petrol. Dials show the number of gallons, number of litres and cost (to two decimal places); while the level of petrol in the tank is seen to vary continuously as it fills, and the tank is graduated from 0 to 1. Several types of multiplicative relationship can therefore be illustrated.

3 Exploring more of the conceptual field of which multiplication and division form a part. In particular, many of the situations we have used lead naturally to proportion (cf Ricco, 1982). This applies certainly to Enlargement, which as yet we have not investigated to any extent.

4 Another aspect we have largely neglected so far is the inverse relationship between multiplication and division.

GENERAL QUESTIONS

Psychological	Pedagogical
What mental processes intervene between the verbal statement of the problem and the symbolic statement of the operation?	What mental processes do we want to occur? How do we bring them about?
How specific/transferable is understanding about the operations learned in one context?	Should explicit awareness of relationships between the various contexts be a teaching aim?

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3D DRAWINGS AND THE PARALLEL CONCEPT

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THE PROBLEM

The problem to be addressed in this paper is: Why do children find it so difficult to represent in their drawings parallel lines in space? Figure 1 shows some examples of the type of errors which frequently occur (Mitchelmore 1976). Figure 1(a) is the most common way of representing a cube among older children (from about 10 yr in England and 15 yr in Jamaica) and, one suspects, among adults without specific training. The other two drawings are typical among 15-yr-old Jamaicans.

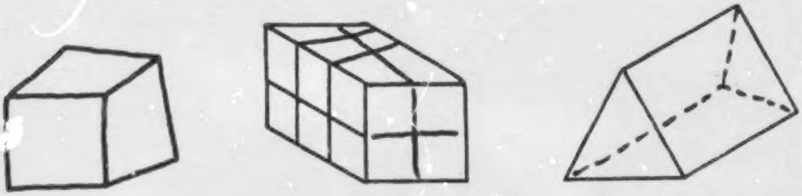


Figure 1

None of the drawings in Figure 1 consistently represents the parallel lines in space by parallel or convergent lines on paper, as is required by all projective models of picture production (Dubery & Willats 1983). Several hypotheses purporting to explain why children who draw in this manner do so will now be examined.

Hypothesis 1 The children do not understand pictorial depth conventions.

The children clearly understand much about pictorial depth representation: (1) A drawing like Figure 1(a) occurs near the end of a developmental sequence in which the use of an oblique line to represent the depth dimension can be seen emerging (Mitchelmore 1978); and (2) all the drawings are topologically correct, that is, they show the correct number of lines and the correct division into regions. The only difference between the drawings in Figure 1 and the projectively correct drawings lies in the direction of the lines.

Hypothesis 2 The children do not recognise the parallel lines on the solid.

There is ample evidence (reviewed by Freeman 1980) that even young children detect parallels in 2D configurations and use them as alignment cues in discrimination and placement tasks. Also, Abravanel (1977) found in his samples of 4-yr-olds in the USA and India that most children thought that a given pair of parallel lines resembled a pair of parallel lines in a different direction more closely than a pair of non-parallel lines in roughly the same direction. In a study recently completed, Freeman and I found that it was very easy to teach the parallel concept to 6-yr-old children in Bristol, England. After 12 examples and counter-examples, children made fewer than 3% errors judging whether 8 pairs of lines were "friends" (defined as lines which went along side by side in the same direction), not significantly different from the error rate in an equivalent control group who were taught that lines were friends if they were drawn in the same colour. However, the error rate rose substantially (to 22%) in a more complex task in which children judged whether a line joining points on the opposite parallel sides of a trapezium was parallel to one or both of the other sides.

It seems possible that older children might experience similar difficulty disembedding the parallel edges of solid figures. An experiment to test this hypothesis directly is in progress.

Hypothesis 2 It is difficult to draw accurately parallel lines which meet an oblique baseline.

It is well-established that young children copy lines which meet an oblique baseline with an error biased towards the perpendicular (Freeman 1980). However, the effect seems to be quite small at the ages we are interested in, and one would expect the presence of a parallel line to improve accuracy. But Mitchelmore (1980) found that the task of drawing the median of a parallelogram, where there are two parallels to help but also two "baselines" to distract, also led to large errors: among 15-yr-old Jamaicans the mean errors were 15.2° for parallelograms with angle 30° and 8.7° for those with angle 60°, compared to 3.8° and 2.6° respectively on the corresponding simple angle-copying tasks (omitting one anomolous orientation where the mean error was away from the perpendicular). And Freeman and I recently found a small but significant bias still present in adults (Bristol University psychology students): a mean bias of 2.3^{o} in drawing the medians of parallelograms with angle 20^{o} and the longest side horizontal.

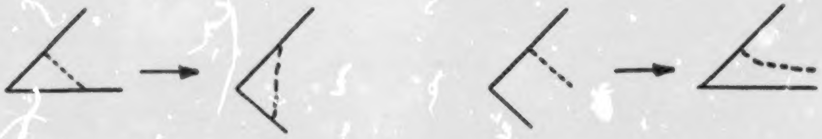
We have further investigated the contribution of parallels and baselines to errors on the median task by administering to the 6-yr-olds in the study mentioned above a set of copying tasks in which the number of parallels and baselines was systematically varied; the baselines were horizontal and the angle was 45^{o}. We found that the smallest mean error was obtained in copying a line parallel to a single line, but that adding or deleting the parallel or adding one or two baselines led to roughly the same mean error (10.8^{o}). Our interpretation is that, in the absence of any baselines, adding a single parallel provides a useful alignment cue but that an additional parallel provides more information than can be processed, so that performance deteriorates to the previous level; and even one baseline causes a similar processing overload.

The perpendicular bias would explain the divergent lines representing the receding edges in Figure 1(a) if one assumes that the square and the middle oblique were drawn first, and Figure 1(c) if the receding hidden edge were drawn first; Figure 1(b) is obvious. A perpendicular bias clearly does exist even in older children, and it would seem likely that the complexity of drawings representing 3D figures is such that children may be unable to make full use of parallel alignment cues to eliminate that bias.

Hypothesis 4 Children do not realise the significance of parallels.

Even if children can recognise parallels, it is possible that they do not spontaneously look for them when inspecting a complex 2D figure to be copied or a 3D figure to be drawn and therefore do not make use of potential alignment cues. To test this possibility in the case of 2D figures, Freeman and I readministered the drawing tasks to our Bristol 6-yr-olds, asking the children to point to the "friends" of each target line before copying it: so half identified parallels (with, this time, about 6% errors) and half identified lines drawn in the same colour. There was no significant difference in mean errors between the two groups or between the two administrations.

We suspect that although 6-yr-old English children can encode the orientation of lines, they do not accept that it is a criterion of an "exact copy". This impression was reinforced incidentally by some unexpected drawings produced by a sizeable minority of 6-yr-olds in another study in which we asked the children to copy a target line after we had changed one of the other lines in the figure. The drawings are shown in Figure 2, in which the broken lines indicate the target lines. The children made topologically exact copies instead of attempting to make the euclidean copies we had intended.



Stimulus response Stimulus response

Figure 2

The same could be true of older children's drawings of 3D figures: they might register the parallels on the solid but not realise their significance for their drawings. A study to test this possibility, using a design similar to that of the 2D study described above, is currently in progress.

Of course, there is a crucial difference between the 2D and 3D cases: an "exact copy" of a 2D figure is congruent to it whereas a "correct drawing" of a 3D figure can never be congruent to the solid - the correspondence between solids and their representations is not even one-to-one. I therefore call copies of 2D figures *isographs* and drawings of 3D figures *homographs*. The criteria for acceptability are different (isographs should be congruent to the stimulus, homographs should represent a view of the stimulus from one position) but they are both conventions which children have to learn. Having learned these conventions, the sense in which parallels are significant is quite different: in isographs, parallels must be preserved, but so also must perpendiculars and all other angles. In homographs, parallels are vital cues and they can either be preserved (as in isometric projection) or rendered convergent (as in perspective), but perpendiculars and other angles must be virtually ignored. In other words, an extra step is required before children can represent 3D parallels as successfully as 2D parallels - a step which it seems very few people take without instruction.

CONCLUSION

To draw an acceptable representation of a regular 3D figure, children must recognise the parallel lines on the solid, realise that parallel lines are significant cues, and then represent the parallels appropriately without succumbing to the perpendicular bias. It appears that weaknesses may appear at each link in this chain. Further research is required to identify the weakest link and to develop procedures for strengthening each one.

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CHILDREN'S CONSTRUCTION OF A NUMBER SYSTEM

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An understanding of number is fundamental to the development of mathematical skills, principles and processes. Children construct an initial concept of number through counting, and the development of this understanding of the oral counting sequence and essential nature of number has been well documented (Gelman and Gallistel, 1978; Fuson, 1980; Steffe, von Glasserfeld and Richards, 1981; Booker, 1982). In addition to the oral number names, they must also learn to read and write the number symbols and to associate these with groups of objects. Those up to ten are learnt in a relatively rote fashion, for there is no inherent reason for assigning either name or symbol to a particular set of objects, but numbers greater than ten follow generative rules based on a system of place value.

Prior to reading and writing symbols for two-digit numbers, children acquire the oral names by a process analogous to their learning of the first ten numbers. An ability to read and write these numbers is then generalised from this knowledge of the number names and counting sequence, and children are readily able to distinguish a number such as 46 from its alternative, 64. Nonetheless, these rules which they impose are not based on place value ideas *per se*, and thus often lead to misunderstandings such as writing 403 for 43 or 70 for 17. Indeed, children's rules tend to be generalisations of the phonic and grammatical rules for language and to be much more complex than the actual numeration concepts they supplant. For, as Gelman and Gallistel (1978) observed, 'historical changes in the pronunciation of the first few count words beyond ten have obscured their derivation from a generative rule'. In particular, it is not until sixteen is reached that a consistent correspondence between the name and symbol emerges and the development of number names for the multiples of ten is similarly obscure.

The difficulties that children experience in learning to read and write two digit numbers are well known, and means of overcoming and avoiding them have been developed (Booker, Irons and Jones, 1980). In particular, the explicit teaching of the part/whole nature of numeration, whereby ten ones form a unit in themselves and the whole number has a tens part and a ones part, is crucial. In an analysis of this development, Ginsburg (1977) suggested that children begin simply with an ability to correctly write (some) numbers, next are able to support a correctly written number over hypothetical alternatives but without explanations, and finally arrive at a stage where numbers are written correctly, discriminated from alternatives and interpreted in terms of place value. He further commented that relatively few children acquire this final level of understanding.

An initial study (Booker, 1982) examined the extent of children's number knowledge on entering school, tracking by tests and follow up interviews the development from the period when formal number work began until the development of the tens and ones part/whole structure. The complexity of the relationships imposed in lieu of a place value system, particularly by the more capable children, pointed to the dangers inherent in delaying the introduction of place value ideas until the second grade or later. Their attempts to extend a knowledge of the early numbers in a way which was sensible to them, led them to adopt a set of 'understandings' with which they felt secure, but which subsequently required a lot of breaking down. Thus, the teacher's job became one of reteaching an understanding of two-digit numbers when it was assumed that the initial teaching of these ideas was occurring.

The responses of April (aged 5 years 1 month) indicate the complexity of the invented rules children can use to impose understanding on the way the number symbols have been put together to form larger numbers. At the first interview she gave the following explanations:

(a) Seventeen is written 17 because the 1 means "teen" and the 7 means "seven". It could not be written as 71 because this is seventy-one where the 7 means "seventy", 1 means "one". Thus, while most numbers were read left-to-right, April had a category she read from right-to-left so as to establish meaning.

(b) In 14 or 64, the 4 means "four", but in 4[illegible] it means "forty" '4 means four when it is at the end, forty when it is at the beginning'.

(c) In 60, 6 means "six", 0 means "ty", 40, 70, 80 and 90 are the same. But in 20, 2 means "twen" while 0 means "ty".

Thus, April had a set of rules for establishing and discriminating all two-digit numbers, but it was certainly not based on a mathematical interpretation of the numbers.

At the second interview, 2 weeks later, the same explanations were given for two-digit numbers, and she was able to give a similar explanation for three-digit numbers.

(a) In "149 the 1 means "one", the 4 means "forty" and the 9 means "nine". She thought it a little strange that the 1 just meant "one", but that was the way it was; it did not mean one hundred, "the hundred comes from nowhere".

(b) She had some difficulty in writing one hundred and six, writing 16 at first, but rewriting it as 106 saying that otherwise it would be sixteeen. The explanation afterwards was an extension of the way zeros were treated in 2-digit numbers. The 1 was for "one", the 0 means "hundred" and the 6 means "six".

(c) In 359, the 3 means "three", the 5 means "fifty" and the nine means "nine". While the hundred "comes from nowhere, you know it is there by looking at the number". Her understanding, then, was based on realisation that three digits in the number gave hundreds, but she was unable to verbalise this.

By the third interview a further 6 weeks later, April was herself able to integrate an awareness of the tens/ones structure into her explanations, even though such an approach had not yet been discussed in class. A stress on placing counters in a 5 x 2 grid and on bundling 10 sticks together had revealed to her a more consistent (mathematical) patterning than the phonic ordering she had previously used.

Several of the other children evolved similar 'explanations' to April's initial descriptions over the first half-semester of year 1, and the following stages became evident in the building up of two-digit number understanding:

1. An ability to write "regular" two-digit numbers such as 42 which fully reflect the place-value system, i.e. there are no zeros or "unusual" tens names. However, if asked whether 24 is the same as 42, the child is unsure, and may sometimes write 24 for 42 or vice versa. Often writes 04, or 40 for fourteen.

2. An ability to write numbers such as 42, but writes 41 as fourteen. Realises that 42 and 24 represent different numbers, but is confused about 41 and 14. Is often able to read back 41 as fourteen, but in another context will read it as forty-one.
3. Fully aware that 24 and 42 are not the same, but without a rational explanation. This understanding extends to include the "unusual" tens such as 23, 34 and 56 as well as knowing how to write and read 40, 60 and 80. Tries to rethink the strategy for fourteen, but may confuse 40 and 14.
4. An ability to write the teen numbers 13-19, often by forming these from right-to-left while the other two-digit numbers are formed from left-to-right.
5. Writing 12 consistently may be the last two-digit number to be grasped because it needs to be seen in context of the other teen numbers despite having a completely dissimilar pronunciation, more in keeping with twenty.

The results of the tasks from this study showed that, in Ginsburg's terminology, children have a highly developed "informal arithmetic" on entering school. Inevitably this informal knowledge is incomplete, yet the apparent depth of understanding it suggests often leads to a rapid development of computational skills when consolidation of a full awareness of number would be more appropriate. Their understanding of the written symbols, and their ability to form these symbols, falls far short of this informal capacity. Earlier suggestions for a teaching sequence and strategies similar to that revealed here have been made on the basis of an analysis of children's difficulties. This study showed that these procedures are rooted in a natural learning sequence, as well as in a task analysis of the learning require and the difficulties that have arisen. It suggests in finer detail the steps that are necessary, and points strongly to the need to match appropriate teaching strategies to the time when children are seeking for the rules and generalisations on which this is based. Indeed, it raises the question whether such an approach might allow children to firmly build a full basis of place value into their conception of two-digit numbers from the outset.

The second study, conducted with both year 1 and year 2 classes from the initial school and an adjacent school of very similar background, was designed to test this proposition. One of the schools was piloting an introduction to number ideas based on the above sequence and treatment (Barry, et al, 1983) while the other followed the traditional treatment; viz 0-10 in year 1, 11-20 in year 2 before the full consideration of two-digit numbers in year 3. The testing and interviewing of the year 1 children from both groups showed a similar range of interpretations to the 1982 sample, although the follow up of development week-by-week was not conducted. Rather, in this study, the year 2 children were given similar tasks so that a measure could be taken of the growth of understanding from year 1 to year 2. Substantial differences occurred between the two groups. While the pilot group showed an understanding of reading and writing two-digit numbers based on a fully integrated use of place value concepts, the traditional group remained at a level of explanation and understanding similar to that of the grade 1 children. Except for two children, explanations were phonic based, supplemented often by recourse to counting strategies of varied sophistication and insight. Many of the traditional group continued to have problems with the 'irregular' tens, to reverse teen numbers, to confuse tens and teens and to be unsure in writing numbers such as 60.

Of particular interest is the treatment of the number formed by reversing the digits in 60. While the majority of children from the pilot group reasoned that this would be the number 6, since it had 0 tens 6 ones, the traditional group predominantly answered that it was not a number 'because numbers can not begin with an oh'. Frequently this group referred to the spelling of numbers, and confusion between 14 and 15, 20 and 12, was brought up because of their similarity of sound. As the study continued, the teacher of this group began to focus on a treatment of place value, but, although many children were able to say how many tens and ones in a particular number when asked, they were unable to use this knowledge to justify the writing or reading of the number.

As a consequence of these investigations, it is possible to describe a hierarchical development in children's construction of a number system to fill out the initial proposals of Ginsburg.

1. Ability to correctly write a number based on phonic rules.
2. Ability to correctly write a number, explaining it by reference to phonic rules and to a knowledge of the counting sequence.
3. Ability to write numbers correctly, knowing when alternatives are wrong, coupled with an ability to interpret numbers in terms of a system of place value.
4. Ability to write numbers correctly knowing why they are correct on the basis of place-value understanding fully integrated into their number concept. Such a fusion is based as much on a visual as on a verbal understanding.

It is the last level that is rarely attained, largely because of the opportunities children are given to construct and cement their own explanations. Indeed, it is the difference between levels 3 and 4 which explains the noted discrepancies between children's capacity to answer numeration questions and the breakdown of their computational procedures due to the misapplication of seemingly known numeration skills and understandings within these procedures.

Application of this sequence in year one classes at several schools in addition to the one where this testing occurred, has tended to show the promised increase in understanding, avoidance of problems and confident growth of new mathematics learning. On the other hand, the lack of mathematical understanding and the complexity as well as the inadequacy of their invented rules provide a fuller picture of the source of so many misconceptions which occur later in computational skill development and when numeration understandings need to be extended to decimal and common fractions. Perhaps at a later stage, children are able to integrate the part/whole structure into their number system, rather than to attach it on to their own 'understanding' of numbers. The present study does not point to this occurring, and a continuing investigation is being undertaken of the year 3 and 4 children in the traditional school to track any changes measured against the school's teaching emphases at those levels.

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STUDIES OF MATHEMATICAL ABILITY

SOME HIGHLIGHTS

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INTRODUCTION

The learning and teaching of mathematics presents an apparently complex picture. The innovative curriculum development projects of the 1960's and 1970's resulted in a change of content but perhaps not a sufficient change of process. Many teachers continued with their same style of teaching and many learners still acquire the same basic misconceptions as their predecessors.

In the U.K. with the new government initiatives (such as NTVEI!) there is now a renewed emphasis on process and the 'negotiated curriculum'.

Much diagnostic work has been carried out, on an international scale, in learners' thinking. The time is ripe for simplification: *Romberg* (1981) for example suggests that for some problem areas in the learning and teaching of mathematics we are now in the model building stage.

I also firmly believe that we now need to look afresh for structure in the field of mathematical abilities. I shall therefore attempt a brief review of such work but will confine attention to studies involving that kind of analysis of data which became unpopular in the 60's and 70's - factor analysis. I cannot resist pointing out that even *Krutetskii* (1976) undertook some factor analysis albeit with due discretion!

Firstly, we shall consider some ideas regarding the nature of general mathematics and then more particularly of school mathematics. The paper in the main will then review some of the studies on mathematical abilities spanning this century: there is no intent to undertake a comprehensive survey! The final discussion will then attempt to pull together some of the findings.

What follows is an outline of ideas to be discussed:

THE NATURE OF MATHEMATICS

- "........the very essence.....is the prevention of waste of the energies of muscle and memory......."
- ".......the process of mathematical discovery is a living and a growing thing....." (*Jourdain* 1879-1919)
- ".......oldest and noblest of man's activities"
- ".......looking for beauty and elegance"...... (*Zeeman* 1983)

These are just a few of the many aspects of the subject as perceived by mathematicians. And what of the content of mathematics? Is it too simplistic to say that general mathematics consists essentially of three areas - Geometry, Algebra and Analysis? Are there common processes involving generalisation, proof, abstraction, logic.......?

Can we then say that school mathematics reflects both the content and processes of more general mathematics?

THE NATURE OF MATHEMATICAL ABILITY

Are there several components to mathematical ability? If so, do these relate to content, to processing, to both?

Hamley (1935) "Mathematical ability is probably a compound of general intelligence, visual imagery, and ability to perceive number and space configurations and to retain such configurations as mental patterns. Each of these could be sub-divided into simpler factors or components."

Krutetskii (1976) identified types of mathematical 'casts of mind': the analytic or abstract type, the geometric type and two modifications of a harmonic type containing features of the other two. Abilities discussed by *Krutetskii* include a striving for clarity and economy, reversibility and curtailment of reasoning.

Bishop (1982) speaking of *Krutetskii's* work says: "The shift in construct from 'ability' to 'abilities' is significant and seems to convey a change from something that differentiates between pupils (high and low ability) to something that they all share in some form, from something which may be inherent, to something which is potentially developable in all pupils."

Hadamard (1950) contrasted 'logical' mathematicians who tend to reason in an abstract and explicitly logical manner with 'intuitive' mathematicians who appear to skip logical steps and often obtain 'instantaneous' insight from internal visuo-spatial models.

FACTOR ANALYTIC STUDIES OF MATHEMATICAL ABILITY

Hamley and most other researchers who took his work as a starting point agreed that the most important component in mathematical ability is the general ability factor 'g'.

Others such as *Barakat* (1950, 1951), *Hamza* (1951, 1952) and *Lee* (1955) claimed to demonstrate an important group factor of 'mathematical ability' over and above 'g'. *Wrigley* (1958) in surveying the earlier work stressed the problem of designing a battery of tests which could legitimately be taken to define 'mathematical ability'. *Wrigley* concluded from his own work that there was "clear and convincing evidence that the four tests of mathematics form a group factor over and above the general factor, and separate from the number factor."

McFarlane Smith (1964) stressed the role of spatial ability. "A comprehensive survey of work on mathematical aptitude indicates that, apart from general intelligence tests, the most predictive tests are also those of the spatial factor. Mechanical arithmetic tests give very little indication of future mathematical or scientific ability."

McCallum, McFarlane Smith & Eliot (1979) add further to this work and conclude...."that the single most important component of mathematical ability is a g/k factor which remains relatively stable in high school years, and which shows little relationship to language comprehension.....".

Bishop (1983) has written a helpful review on the interface between the visual/spatial field and mathematics education.

DeGuire (1980) has undertaken the difficult task of attempting to classify the considerable factor analytic literature on mathematical abilities.

The work of *Furneaux, Rees, Cornyn* and *Barr* (1975, '76, '77, '78 and '80) represents a departure from the method of other researchers and will be discussed. Studies of learners of all ages suggest that within mathematics itself there are dimensions of ability.

MATHEMATICAL ABILITY AND OTHER ISSUES

Any discussion of mathematical abilities will be concerned amongst other issues with

- the debate over 'nature and nurture' (*Vernon* 1965, *Krutetskii* 1976 and *Blakemore* 1983),
- the hierarchical nature of abilities (*Cattell* 1963 and *Vernon* 1965),
- sex differences (these are well documented).

All these will be briefly discussed.

CONCLUDING DISCUSSION

An attempt will be made to 'pull the threads' together and see if a clearer picture of the nature of mathematical ability is possible. I ask, however, for readers' tolerance: a 'snapshot view' cannot do justice to the work available. I do hope however that this paper will provide food for thought. To aid this further the paper will end with a comparison between the work of one of the first researchers in this field, *Rogers* (1918) with some current work.

In 1965 P.E. *Vernon* said: "But I would not claim to have done more than to have scratched the surface and to have raised many more problems than I have solved."

I hope that in 1983 we have come a little further.

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